Part II Problems and Solutions

Problem 1: *[ODEs via Laplace transform] Let a and b be real numbers, with $a \neq 0$.*

(a) *Find the unit impulse response and unit step response for the first order operator $aD + bI$ by using Laplace transform methods to solve initial value problems with rest initial conditions.*

(b) *Solve $a\dot{x} + bx = t\,u(t)$ with rest initial conditions in three ways.*
(i) Undetermined coefficients to get x_p, and add the appropriate transient.
(ii) Compute $w(t) * t$ (using the value for $w(t)$ you found in **(a)***).*
(iii) Apply Laplace transform, solve, and transform back.

Solution: **(a)** The unit impulse response is the solution to

$$a\dot{x} + bx = \delta(t), \quad \text{with rest IC.}$$

Taking the Laplace transform of this equation gives

$$asX(s) + bX(s) = 1.$$

Solving for $X(s)$ and then taking the inverse Laplace transform we get

$$X(s) = \frac{1}{as+b} = \frac{1/a}{s + (b/a)} \Rightarrow x(t) = \frac{1}{a}u(t)e^{-bt/a}.$$

The unit impulse response is $w(t) = \dfrac{1}{a}u(t)e^{-bt/a}$.

The unit step response is the solution to

$$a\dot{x} + bx = u(t), \quad \text{with rest IC.}$$

Laplace transform: $(as+b)X(s) = \dfrac{1}{s} \Rightarrow X(s) = \dfrac{1}{s(as+b)}$.

Partial fractions: $\dfrac{1}{s(as+b)} = \dfrac{A}{s} + \dfrac{B}{as+b}$.

Coverup method: $A = 1/b$, $B = -a/b$.

Unit step response (via Laplace inverse): $x(t) = u(t)\left(\dfrac{1}{b} - \dfrac{1}{b}e^{-bt/a}\right)$.

(b) This is called the "unit ramp response."

(i) Case 1, $b \neq 0$: For $t > 0$, try the solution $x_p = c_1 t + c_0$.
Substitute into the DE and solve for c_1 and c_2:

$$ac_1 + b(c_1 t + c_0) = t \Rightarrow bc_1 = 1,\ ac_1 + bc_0 = 0 \Rightarrow c_1 = \frac{1}{b},\ c_0 = -a/b^2.$$

Particular solution: $x_p = \dfrac{1}{b}t - \dfrac{a}{b^2}$.
General solution: $x(t) = x_p + ce^{-bt/a}$
Rest IC: $x(0) = x_p(0) + c = -\dfrac{a}{b^2} + c \Rightarrow c = \dfrac{a}{b^2}$.
Solution (for all t): $x(t) = u(t)\left(\dfrac{1}{b}\,t - \dfrac{a}{b^2} + \dfrac{a}{b^2}\,e^{-bt/a}\right)$.

Case 2, $b = 0$: In this case $a\dot{x} = t$, which has general solution $x(t) = \frac{1}{2a}t^2 + c$. The rest initial conditions imply $0 = x(0) = c$, so $x(t) = u(t)\frac{1}{2a}t^2$.

(ii) If $b \neq 0$: $w(t) * t = \displaystyle\int_0^t \frac{1}{a}e^{-b(t-\tau)/a}\tau\, d\tau = \frac{1}{a}e^{-bt/a}\int_0^t e^{b\tau/a}\tau\, d\tau$. Do this by parts: $u = \tau$, $du = d\tau$, $dv = e^{b\tau/a}d\tau$, $v = \frac{a}{b}e^{b\tau/a}$, $w(t) * t =$
$\displaystyle\frac{1}{a}e^{-bt/a}\left(\tau\frac{a}{b}e^{b\tau/a}\Big|_0^t - \int_0^t \frac{a}{b}e^{b\tau/a}\, d\tau\right) = \frac{1}{a}e^{-tb/a}\left(t\frac{a}{b}e^{bt/a} - \frac{a^2}{b^2}(e^{bt/a} - 1)\right) = \frac{1}{b}t - \frac{a}{b^2}(1 - e^{-bt/a})$.

If $b = 0$, $w(t) * t = \displaystyle\int_0^t \frac{1}{a}\tau\, d\tau = \frac{1}{a}\frac{t^2}{2}$.

(iii) $a\dot{x} + bx = t$ has Laplace transform $asX + bX = \dfrac{1}{s^2}$, so

$$X = \frac{1}{s^2(as+b)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{as+b}$$

Coverup: Multiply by s^2 and set $s = 0$ to get $B = \dfrac{1}{b}$.

Multiply by $as + b$ and set $s = -\dfrac{b}{a}$ to get $C = \dfrac{a^2}{b^2}$.

Here's a clean way to get A: multiply through by s and then take s very large in size. You find $0 = A + \dfrac{C}{a}$, or $A = -\dfrac{a}{b^2}$.

So $X = -\dfrac{a/b^2}{s} + \dfrac{1/b}{s^2} + \dfrac{a/b^2}{s + b/a}$, which is the Laplace transform of $x = -\dfrac{a}{b^2} + \dfrac{1}{b}t + \dfrac{a}{b^2}e^{-bt/a}$.

If $b = 0$, $a\dot{x} = t$ has Laplace transform $(as)\,X = \dfrac{1}{s^2}$ so $X = \dfrac{1}{a}\dfrac{1}{s^3}$, and thus $x = u(t)\dfrac{1}{a}\dfrac{1}{2}t^2$