Definition of Laplace Transform

1. Definition of Laplace Transform

The Laplace transform of a function $f(t)$ of a real variable t is another function depending on a new variable s, which is in general complex. We will denote the Laplace transform of f by $\mathcal{L}f$. It is defined by the integral

$$(\mathcal{L}f)(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt, \tag{1}$$

for all values of s for which the integral converges.

There are a few things to note.

- $\mathcal{L}f$ is only defined for those values of s for which the improper integral on the right-hand side of (1) converges.
- We will allow s to be complex.
- As with convolution the use of 0^-, in the definition (1) is necessary to accomodate generalized functions containing $\delta(t)$. Many textbooks do not do this carefully, and hence their definition of the Laplace transform is not consistent with the properties they assert. In those cases where 0^- isn't needed we will use the less precise form

$$(\mathcal{L}f)(s) = \int_{0}^{\infty} f(t)e^{-st}\,dt. \tag{1'}$$

- Also, as with convolution, the limits of integration mean that the Laplace transform is only concerned with functions on $(0^-, \infty)$.

2. Notation, $F(s)$

We will adopt the following conventions:

1. Writing $(\mathcal{L}f)(s)$ can be cumbersome so we will often use an uppercase letter to indicate the Laplace transform of the corresponding lowercase function:

   $$(\mathcal{L}f)(s) = F(s), \qquad (\mathcal{L}g)(s) = G(s), \text{ etc.}$$

   For example, in the formula

   $$\mathcal{L}(f') = sF(s) - f(0^-)$$

   it is understood that we mean $F(s) = \mathcal{L}(f)$.

2. If our function doesn't have a name we will use the formula instead. For example, the Laplace transform of the function t^2 is written $\mathcal{L}(t^2)(s)$ or more simply $\mathcal{L}(t^2)$.

3. If in some context we need to modify $f(t)$, e.g. by applying a translation by a number a, we can write $\mathcal{L}(f(t-a))$ for the Laplace transform of this translation of f.

4. You've already seen several different ways to use parentheses. Sometimes we will even drop them altogether. So, if $f(t) = t^2$ then the following all mean the same thing

$$(\mathcal{L}f)(s) = F(s) = \mathcal{L}f(s) = \mathcal{L}(f(t))(s) = \mathcal{L}(t^2)(s); \quad \mathcal{L}f = F = \mathcal{L}(t^2).$$

3. First Examples

For the first few examples we will explicitly use a limit for the improper integral. Soon we will do this implicitly without comment.

Example 1. Let $f(t) = 1$, find $F(s) = \mathcal{L}f(s)$.

Solution. Using the definition (1′) we have

$$\mathcal{L}(1) = F(s) = \int_0^\infty e^{-st}\,dt = \lim_{T\to\infty} \frac{e^{-st}}{-s}\bigg]_0^T = \lim_{T\to\infty} \frac{e^{-sT}-1}{-s}\bigg]_0^T.$$

The limit depends on whether s is positive or negative.

$$\lim_{T\to\infty} e^{-sT} = \begin{cases} 0 & \text{if } s > 0 \\ \infty & \text{if } s < 0. \end{cases}$$

Therefore,

$$\mathcal{L}(1) = F(s) = \begin{cases} \frac{1}{s} & \text{if } s > 0 \\ \text{diverges} & \text{if } s \le 0. \end{cases}$$

(We didn't actually compute the case $s = 0$, but it is easy to see it diverges.)

Example 2. Compute $\mathcal{L}(e^{at})$.

Solution. Using the definition (1′) we have

$$\mathcal{L}(e^{at}) = \int_0^\infty e^{at}e^{-st}\,dt = \lim_{T\to\infty} \frac{e^{(a-s)t}}{a-s}\bigg]_0^T = \lim_{T\to\infty} \frac{e^{(a-s)T}-1}{a-s}\bigg]_0^T.$$

The limit depends on whether $s > a$ or $s < a$.

$$\lim_{T\to\infty} e^{(a-s)T} = \begin{cases} 0 & \text{if } s > a \\ \infty & \text{if } s < a. \end{cases}$$

Therefore,

$$\mathcal{L}(e^{at}) = \begin{cases} \frac{1}{s-a} & \text{if } s > a \\ \text{diverges} & \text{if } s \leq a. \end{cases}$$

(We didn't actually compute the case $s = a$, but it is easy to see it diverges.)

We have the first two entries in our table of Laplace transforms:

$$\begin{aligned} f(t) = 1 \quad &\Rightarrow \quad F(s) = 1/s, \qquad && s > 0 \\ f(t) = e^{at} \quad &\Rightarrow \quad F(s) = 1/(s-a), && s > a. \end{aligned}$$

4. Linearity

You will not be surprised to learn that the Laplace transform is linear. For functions f, g and constants c_1, c_2

$$\mathcal{L}(c_1 f + c_2 g) = c_1\mathcal{L}(f) + c_2\mathcal{L}(g)$$

This is clear from the definition (1) of $\mathcal{L}$ and the linearity of integration.