Part I Problems and Solutions

Problem 1: *Let z be a given complex number. From the definition of the Laplace transform, find $\mathcal{L}\left(e^{-zt}\right)$ and also its region of convergence.*

Solution:

$$\begin{aligned}
\mathcal{L}\left(e^{-zt}\right) &= \int_0^\infty e^{-zt} dt \\
&= \int_0^\infty e^{-(z+s)t} dt = -\frac{1}{z+s} e^{-(z+s)t}\bigg|_0^\infty \\
&= -\frac{1}{z+s}\left(e^{-(z+s)\infty} - e^{-(z+s)0}\right) \\
&= \frac{1}{z+s}
\end{aligned}$$

The region of convergence is all s for which $e^{-(z+s)(\infty)} = \lim_{m\to\infty} e^{-(z+s)m} = 0$; that is, all s for which $\mathrm{Re}(z+s) > 0$, or $\mathrm{Re}(s) > -\mathrm{Re}(z)$.

Problem 2: *By using the table of formulas, find:*

(a) $\mathcal{L}\left(e^{-t}\sin 3t\right)$ *(b)* $\mathcal{L}\left(e^{2t}(t^2 - 3t + 2)\right)$.

Solution: (a) $\mathcal{L}(\sin 3t) = \frac{3}{s^2+9} = F(s)$. By the exponential shift rule,

$$\mathcal{L}\left(e^{-t}\sin 3t\right) = F(s+1) = \frac{3}{(s+1)^2+9}$$

(b) $\mathcal{L}\left(t^2 - 3t + 2\right) = \frac{2}{s^3} - \frac{3}{s^2} + \frac{2}{s} = F(s)$

By exponential shift rule,

$$\mathcal{L}\left(e^{2t}(t^2 - 3t + 2)\right) = F(s-2) = \frac{2}{(s-2)^3} - \frac{3}{(s-2)^2} + \frac{2}{s-2}$$

Problem 3: *Find $\mathcal{L}\left(e^{-t}\sin 3t\right)$ by writing $e^{-t}\sin 3t$ as a linear combination of complex exponentials. Compare the answer to that obtained in the previous problem.*

Solution:

$$
\begin{aligned}
\sin 3t &= \frac{1}{2i}\left(e^{3it} - e^{-3it}\right) \\
e^{-t}\sin 3t &= \frac{1}{2i}\left(e^{-(1-3i)t} - e^{-(1+3i)t}\right) \\
\mathcal{L}\left(e^{-t}\sin 3t\right) &= \frac{1}{2i}\left(\mathcal{L}\left(e^{-(1-3i)t}\right) - \mathcal{L}\left(e^{-(1+3i)t}\right)\right) \quad = \frac{1}{2i}\left(\frac{1}{s+1-3i} - \frac{1}{s+1+3i}\right) \\
&= \frac{1}{2i}\left(\frac{(s+1+3i)-(s+1-3i)}{(s+1)^2+9}\right) \\
&= \frac{3}{(s+1)^2+9}
\end{aligned}
$$

This is the same as previously found.