18.03SC Differential Equations, Fall 2011
Transcript – Computing Fourier Series

PROFESSOR: Hi, everyone. Welcome back. So today we'd like to tackle a problem in Fourier series. And specifically, we're just going to compute the Fourier series for a simple function. So the function we're interested in is f of t, which we're told is periodic with period 2pi. F of t is 1 from minus pi to 0, and then it's minus 1 from 0 to pi.

So first off, we're interested in sketching f of t. Secondly, we'd like to compute the Fourier series for f of t. And then thirdly, we'd like to sketch the first non-0 term of the Fourier series. And we can specifically sketch this single term on top of f of t. So I'll let you think about this problem for now, and I'll be back in a moment.

Hi, everyone. Welcome back. So let's take a look at sketching f of t. So for part a, we have our axes, t. And we're told f of t within some interval. So we might as well plot f of t on that interval. So minus pi, pi and 0, we know that f of t is 1 from minus pi to 0. We're also told that it's minus 1 from 0 to pi.

And now to fill in the blanks or to complete the picture of f, we're told that it has a period of 2pi. So note that they've told us what f looks like over the range of minus pi to pi, which is the length of 2pi. So basically what we can do is we can use this as a stamp and just pick up this entire picture, shift it over one period 2pi, and just thinking of this picture in stamping it in multiple places. So just filling this in it's going to look like a square wave, which jumps between minus 1 and 1 at every multiple of pi. So this concludes part a.

For part b, which is the real meat of the problem, we're interested in computing a Fourier series for f of t. Now, we can always write down a Fourier series for any periodic function. And specifically in this case, for part b, the periodic function we're interested in has period 2pi. So for the class notes, we've identified L with half the period. So in this case, L is 2pi divided by 2, which gives us pi.

And just to recall what a Fourier series is, what we do is we try and take our function f of t and write it down as a summation of sines and cosines. So in this case for function f of t, which is 2pi periodic, it's going to look something like this. It's going to a of 0 plus sum from n equals 1. And there's going to be infinitely many terms, but in this case we have a of n times cosine of nt. And it's nt here because we have period 2pi. Plus bn sin nt. So this is the general form.

And when asked to compute the Fourier series of a function, the main difficulty is to compute these coefficients an and bn. However, that essentially boils down to working out some integrals.

So let's take a look at what a of 0 is. So the formula for a0 is 1 over 2L. So in this case, it's 1 over 2pi, times the integral over 1 period of the function, from minus pi to pi, of just f of t. So notice how a0 is just the average of the function.

So if we take a look at the function f of t, f of t spends exactly half of its time at 1 and half of its time at minus 1. So immediately we could guess that the average value of f of t is going to be 0. If you wanted to work it out specifically, we would have 1 over 2pi minus pi to 0, f of t takes on the value of plus 1. And then from 0 to pi, f of t takes on the value of minus 1. So we would end up getting pi minus pi, which is 0.

For an, the formula is 1 over half the period. So note how a of 0 is just a special case. We always have the full period in a0, but in an and bn, the factor that divides the integral is always going to be half the period times minus pi to pi f of t cosine nt dt.

And I should point out that, in general, we only need to integrate over one period of the function. So in some sense there's nothing special about minus pi and pi. It's just very often these are the easiest bounds of integration to integrate over. But in practice, we could have used 0 to 2pi or any other interval, as long as it's exactly one period of the function.

So in this case, I'd just like to take a look at the symmetry of f of t. And we note that the function f of t is actually odd about the origin. So if f of t is odd and cosine t is an even function, then an odd times an even function is going to be an odd function. And when you integrate an odd function from minus any value to the same positive value, so in this case minus pi to pi, we always get 0. So this is actually 0, because we're integrating an odd function over a symmetric interval.

So lastly, we have the values of bn, which are 1 over pi minus pi to pi f of t of sine nt dt. And if we were to look at just the symmetry argument again, f of t is an odd function, sine t is an odd function, an odd times an odd function is an even function. When you integrate an even function over a symmetric bound, you will essentially get twice the value of the integral from 0 to one of the bounds. So b of n in this case doesn't vanish, which means we actually have to do some work.

So what we do? Well, we know the value of f of t on two intervals, so we're just going to have to work out each interval. Minus pi is 0, it takes on the value of 1. So we have sine nt. And then from 0 to pi, f of t takes on the value of minus 1 sine nt dt.

And you'll note that these integrals are actually the same. So this is negative 2 over pi zero to pi sine nt dt, which if we integrate is negative 1 over n cosine nt evaluated between 0 and pi. So if I work this out, we get minus and a minus, minus 1 over n cosine n pi plus 1 over n. So note that cosine of 0 is just 1.

And now if we take a look at cosine n pi, we see that cosine n pi oscillates between minus 1 and 1. So cosine of pi is negative 1, cosine of 2pi is 1, cosine of 3pi is minus 1. So this term right here is actually negative 1 to the n. So we have 2 over n pi 1 minus negative 1 to the n.

And now if we just plug in some values of b of 1, b of 2, b of 3, b of 4, we can see what pattern merges in the bs. So b of 1, if I plug in 1, I get 1 minus negative 1. It's going to be 2. So I get minus 4 over pi. b of 2 it's going to be 1 minus minus 1 squared is just 1. So this vanishes. b of 3 is going to be 1 minus minus 1 cubed, which is negative 1. So again, we get negative 4 over 3pi. b of 4 is going to be 0.

So it's sometimes useful the write out what the Fourier series looks like. So I'll just write it out right here. So we have f of t is going to be negative 4 over pi times sine of t plus 1/3 sine of 3t plus 1/5 sine of 5t plus dot, dot, dot. So this concludes part b.

And now lastly, for part c, we're asked to sketch what does the first Fourier term look like. So in this case, the first Fourier term is going to be negative 4 over pi times sine t. So I'm going to go back to our diagram from part a. So let's go back to our diagram from part a.

Now what is minus 4 over pi sine t look like? Well, it's a sine wave that has exactly period 2pi, and it's going to line up exactly with this square wave. In addition, minus 4 over pi is just slightly larger than 1. So we're going to end up with sin, which peaks just slightly above 1 and slightly below 1. It's going to go through 0, and it's going to go through each multiple of pi. So it might look something like this.

So this is the first Fourier term in the series. And notice how this first Fourier term is actually pretty good approximation to the square wave, considering it's just one term in a series. As we add more terms in the series, we're going to get something which looks closer and closer to a square wave function.

So I'd just like to quickly recap. When computing the Fourier series for a periodic function, the first useful thing to do is just write down the formula for a Fourier series, and then write down the formulas for the coefficients of the Fourier series. So write down the formulas for a0 an, bn.

When computing a0, you can often just look at the average of the function. When computing an and bn, it's also useful look at the symmetry of your function. And if it's either even or odd symmetric then typically, either all the ans or all the bns will vanish. And then when you work at the integrals, you can then reconstruct the Fourier series.

So I would like to conclude here, and I'll see you next time.

So I'd like now to take a look at part c. And in part c, we have the differential equation x dot dot plus 6x dot plus 4x equals f not cosine omega t. And again, the amplitude response is going to equal 1 over the absolute value of p of i omega. And in this case, p of i omega is going to be 1 over-- Well, we still have the 4 minus omega squared term. Instead of x dot, we now have 6x dot, which gives us 6i omega.

And then again, we want to take the absolute value of this complex number. And when we take the absolute value, we just get the sum of the real parts squared plus the sum of the imaginary parts squared, which in this case is going to be 36 omega squared, the whole quantity squared rooted, and then we have 1 over this value.

So now if we'd like to plot this function, we can still do the same trick and try to maximize or find the critical points of the denominator under the radical. And if we did this, in this case we would find that the only critical point is when omega is equal to 0. Secondly, if we look at omega going to infinity, we see that the denominator goes to infinity. So this whole quantity must go to 0.

So if I were to go back here to the amplitude response for part c, again, when omega is equal to 0 it's going to start off at 1/4. I've just argued that it goes to 0 as omega goes to infinity. And since there are no critical points, we must smoothly paste the function between the two. And in fact, it's always decreasing. So the amplitude response, in this case, is just a decreasing function.

So this concludes part c. And now I'll take a look at part d. Discuss the resonance for each system. So in part a, we had no damping. And we saw that there was a resonance at omega equals 2. And the resonance manifested itself in the amplitude response graph with a divergent asymptote at omega is equal to 2. So as you drive the system close to omega equals 2, the amplitude of the system starts to diverge.

In case two we introduced damping into the system. So we still have a very large amplitude response at omega equals the square root of 7/2, however it's no longer infinite. And then lastly, when we increased damping even further so we had the 6x dot term, the presence of a peak disappeared. And in fact, the amplitude response just monotonically decayed from 1/2 to infinity. So just constantly decreased to 0.

So I'd just like to conclude there, and I'll see you next time.