18.03SC Unit 3 Practice Exam and Solutions

Study Guide on Step, Delta, Convolution, Laplace

You can think of the step function $u(t)$ as any nice smooth function which is 0 for $t < -a$ and 1 for $t > a$, where a is a positive number which is much smaller than any time scale we care about in the context we are studying at the moment. Similarly, the best way for you to understand the "delta function" is to think of it as any smooth function which is zero except in the immediate neighborhood of $t = 0$ and which has integral 1.

So $\dot{u}(t-b) = \delta(t-b)$ and if $a < b$ then $\int_a^t \delta(\tau - b)\, d\tau = u(t-b)$.

A function $f(t)$ is "regular" or "piecewise smooth" if it can be broken into pieces each having all higher derivatives and such that at each breakpoint $f^{(n)}(a-)$ and $f^{(n)}(a+)$ exist. A "singularity function" is a linear combination of shifted delta functions. A "generalized function" $f(t)$ is a sum $f(t) = f_r(t) + f_s(t)$ of a regular function and a singularity function. Any regular function $f(t)$ has a "generalized derivative" $f'(t)$, with regular part $f_r'(t)$ the regular derivative of $f(t)$ whereever it exists, and singular part $f_s'(t)$ given by a sum of terms $(f(a+) - f(a-))\delta(t-a)$, one for each break in the graph of $f(t)$.

Then $\int_a^c f'(t)\, dt = f(c) - f(a)$. (To be more precise, $\int_{a-}^{c+} f'(t)\, dt = f(c+) - f(a-)$.)

For the rest of this unit, all "signals" (functions of t) are supposed to be zero for $t < 0$. We look for solutions to the differential equation $p(D)x = q(t)$, especially with "rest initial conditions," so that $x(t) = 0$ for $t < 0$ and x has as many derivatives as possible.

The *unit impulse response* or *weight function* of the operator $p(D)$ is the solution $w(t)$ to the equation $p(D)w = \delta(t)$, with rest initial conditions. If $p(s) = a_n s^n + \cdots + a_0$ with $a_n \neq 0$, and x is such that $p(D)x = 0$ and $x(0) = \cdots = x^{(n-2)}(0) = 0$ and $x^{(n-1)}(0) = \frac{1}{a_n}$, then $w(t) = u(t)x(t)$. The *unit step response* is the solution to $p(D)v = u(t)$ with rest initial conditions; for $t > 0$ this coincides with the solution to $p(D)x = 1$ such that $x(0) = \cdots = x^{(n-1)}(0) = 0$. Since $\dot{u}(t) = \delta(t)$, $\dot{v}(t) = w(t)$.

The response to a unit impulse determines the system response to any other input signal (with rest initial conditions): the solution to $p(D)x = q(t)$ is given by $x(t) = w(t) * q(t)$, where the asterisk indicates the *convolution product* $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)\, d\tau$.

The convolution product has properties analogous to the ordinary product: $f * (g * h) = (f * g) * h$, $f * (ag + bh) = a(f * g) + b(f * h)$, $f * g = g * f$. Also $f(t) * \delta(t) = f(t)$. If you feed the output of a system (with unit impulse response $g(t)$) into another system (with unit impulse response $f(t)$), you get a "composite system" with unit impulse response $f(t) * g(t)$.

The Laplace transform carries a generalized function (with $f(t) = 0$ for $t < 0$) to a function $F(s)$ of a complex variable s. It obeys a bunch of rules, the most important of which are *linearity* and the *t-derivative* rule $\mathcal{L}[f'(t)] = sF(s)$, where here $f'(t)$ denotes the generalized derivative. There are standard computations, too, including $\mathcal{L}[\delta(t)] = 1$ (the constant function of s). These already imply that the unit impulse response $w(t)$ of $p(D)$ satisfies $\mathcal{L}[w(t)] = \dfrac{1}{p(s)}$. $W(s) = \mathcal{L}[w(t)]$ is the *transfer function* of the operator $p(D)$. Also the solution to $p(D)x = f(t)$ with rest initial conditions has Laplace transform $X(s) = W(s)F(s)$.

This relates to the formula $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$.

If $f(t)$ and $g(t)$ have the same Laplace transform, then $f(a-) = g(a-)$ and $f(a+) = g(a+)$ for every $a \geq 0$.

If $F(s)$ is a rational function (quotient of a polynomial by a polynomial), "continued fractions" provides standard way to find $f(t)$. "Coverup" is an efficient way to compute some of the coefficietns. This couples with completing the square and the s-shift rule.

The *pole diagram* of $F(s)$ is the set of complex numbers s at which $|F(s)|$ becomes infinite. This is the set of uncancelled zeros in its denominator. The pole diagram of $F(s) = \mathcal{L}[f(t)]$ determines the region of convergence of the integral: it is the region to the right of the vertical line through the rightmost pole. The pole diagram also controls much of the behavior of $f(t)$ for large t (while saying nothing about behavior for small t). The rightmost poles dominate. A pole at $a + i\omega$ leads to exponential growth/decay like e^{at} (or some polynomial times e^{at}) and oscillation of circular frequency ω.

The transfer function occured earlier in the course: the exponential solution to $p(D)z = e^{rt}$ is $z_p = W(r)e^{rt}$. So the sinusoidal solution to $p(D)x = \cos(\omega t)$ is $x_p = |W(i\omega)|\cos(\omega t - \phi)$, where $\phi = -\mathrm{Arg}\,(W(i\omega))$: $W(i\omega)$ is the complex gain if e^{rt} itself is regarded as the input signal), and $|W(i\omega)|$ is the gain. The amplitude response curve is obtained by intersecting the graph of $|W(s)|$ with the plane above the imaginary axis.

Practice Hour Exam

1. Let ω be a positive constant. We drive a harmonic oscillator with a square wave of circular frequency ω: $\ddot{x} + 4x = \mathrm{sq}(\omega t)$.

(a) Write down a periodic solution to the equation, if ω is such that there is one.

(b) For what values of ω does there fail to be a periodic solution?

2. Let $f(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 - t & \text{for} \quad 0 < t < 1 \\ 1 & \text{for} \quad t > 1 \end{cases}$.

(a) Sketch the graph of $f(t)$.

(b) Sketch the graph of the generalized derivative $f'(t)$.

(c) Write down a formula for $f'(t)$ in terms of step and delta functions.

3. (a) Compute the convolution product $t * t^6$.

(b) A certain operator $p(D)$ has unit impulse response $w(t) = 2u(t)te^{-t}$. What is the solution to $p(D)x = e^{-t}$ with rest initial conditions?

4. (a) What is the Laplace transform of the solution to the equation $\ddot{x} + 2\dot{x} + 2x = 1$ having rest initial conditions?

(b) What function $f(t)$ has Laplace transform $F(s) = \dfrac{2s}{(s+1)(s^2+2s+5)}$?

5. (a) Sketch the pole diagram for the function $F(s) = \dfrac{2}{(s+1)(s^2+2s+5)}$.

(b) Give an example of a function $f(t)$ whose Laplace transform has poles at $s = 2$ and $s = -3 \pm 4i$ and nowhere else.

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}\,dt$ for $\mathrm{Re}\, s \gg 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s)$, where $f'(t)$ denotes the generalized derivative.

$\mathcal{L}[f_r'(t)] = sF(s) - f(0+)$ if $f(t)$ is continuous for $t > 0$.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2} \qquad \mathcal{L}[t\sin(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Fourier coefficients for periodic functions of period 2π:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\mathrm{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\mathrm{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$

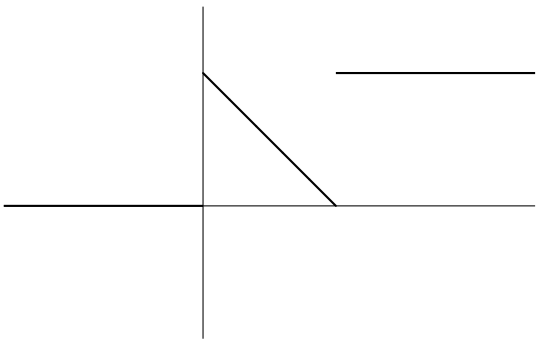
Solutions

1. (a) $\mathrm{sq}(\omega t) = \frac{4}{\pi}\left(\sin(\omega t) + \frac{\sin(3\omega t)}{3} + \frac{\sin(5\omega t)}{5} + \cdots\right)$. Using superposition and the fact that $\ddot{x} + 4x = A\sin(\omega t)$ has solution $x_p = A\frac{\sin(\omega t)}{4-\omega^2}$,

$$x_p = \frac{4}{\pi}\left(\frac{\sin(\omega t)}{4-\omega^2} + \frac{\sin(3\omega t)}{4-9\omega^2} + \frac{\sin(5\omega t)}{4-25\omega^2} + \cdots\right)$$

(b) This solution does not exist if $\omega = 2/(\text{odd integer})$.

2. (a)



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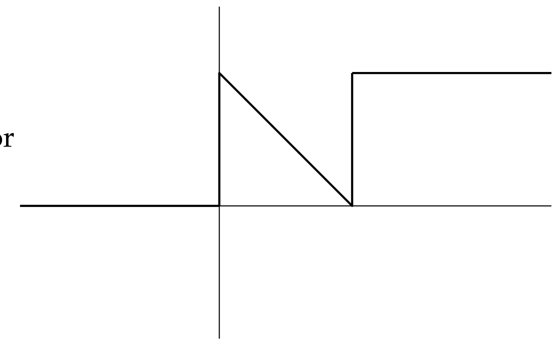
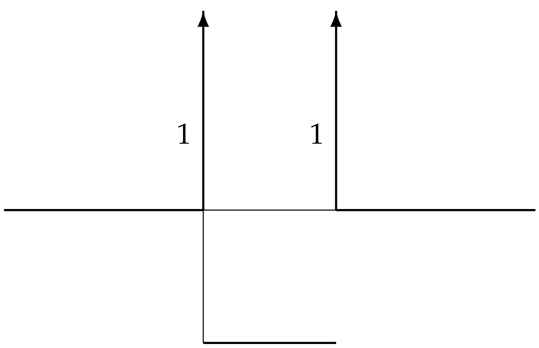


2. (b)



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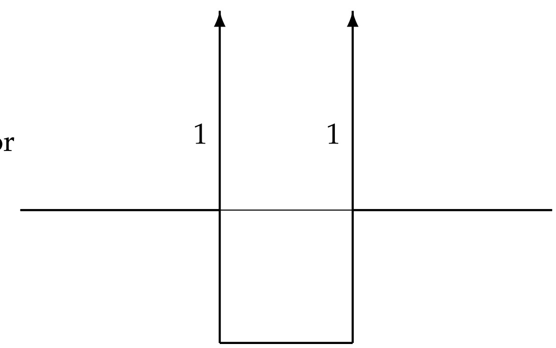

(c) $f'(t) = -(u(t) - u(t-1)) + \delta(t) + \delta(t-1)$.

3. (a) $t * t^6 = \int_0^t (t-\tau)\tau^6\, d\tau = \int_0^t (t\tau^6 - \tau^7)\, d\tau = \left[t\frac{\tau^7}{7} - \frac{\tau^8}{8}\right]_0^t = t^8\left(\frac{1}{7} - \frac{1}{8}\right) = \frac{t^8}{56}$.

(b) $x(t) = w(t) * e^{-t} = \int_0^t 2(t-\tau)e^{-(t-\tau)}e^{-\tau}\, d\tau = e^{-t}\int_0^t 2(t-\tau)\, d\tau = e^{-t}\left[-(t-\tau)^2\right]_0^t = t^2e^{-t}$

4. (a) $s^2X + 2sX + 2X = \frac{1}{s}$, so $X = \frac{1}{s(s^2+2s+2)}$.

(b) $\frac{2s}{(s+1)(s^2+2s+5)} = \frac{a}{s+1} + \frac{b(s+1)+c}{(s+1)^2+4}$.

By coverup, $a = \frac{2(-1)}{(-1)^2 + 2(-1) + 5} = -\frac{1}{2}$, $b(2i) + c = \frac{2(-1+2i)}{2i} = 2 + i$ so $b = \frac{1}{2}$ and $c = 2$. Thus $f(t) = -\frac{1}{2}e^{-t} + e^{-t}\left(\frac{1}{2}\cos(2t) + \sin(2t)\right)$.

5. (a) Poles at $s = -1$ and at $s = -1 \pm 2i$.

(b) $e^{2t} + e^{-3t}\sin(4t)$, or many others.