18.03SC Unit 3 Exam

1. A certain periodic function has Fourier series

$$f(t) = 1 + \frac{\cos(\pi t)}{2} + \frac{\cos(2\pi t)}{4} + \frac{\cos(3\pi t)}{8} + \frac{\cos(4\pi t)}{16} + \cdots$$

(a) What is the minimal period of $f(t)$? [4]

(b) Is $f(t)$ even, odd, neither, or both? [4]

(c) Please give the Fourier series of a periodic solution (if one exists) of [8]

$$\ddot{x} + \omega_n^2 x = f(t)$$

(d) For what values of ω_n is there no periodic solution? [4]

2. Let $f(t) = (u(t+1) - u(t-1))t$.

(a) Sketch a graph of $f(t)$. [6]

(b) Sketch a graph of the generalized derivative $f'(t)$. [6]

(c) Write a formula for the generalized derivative $f'(t)$. [8]

3. Let $p(D)$ be the operator whose unit impulse response is given by $w(t) = e^{-t} - e^{-3t}$.

(a) Using convolution, find the unit step response of this operator: the solution to $p(D)v = u(t)$ with rest initial conditions. [10]

(b) What is the transfer function $W(s)$ of the operator $p(D)$? [5]

(c) What is the characteristic polynomial $p(s)$? [5]

4 (a) Find a generalized function $f(t)$ with Laplace transform $F(s) = \dfrac{e^{-s}(s-1)}{s}$. [10]

(b) Find a function $f(t)$ with Laplace transform $F(s) = \dfrac{s+10}{s^3+2s^2+10s}$. [10]

5. Let $W(s) = \dfrac{s+10}{s^3 + 2s^2 + 10s}$.

(a) Sketch the pole diagram of $W(s)$. [10]

(b) If $W(s)$ is the transfer function of an LTI system , what is the Laplace transform of the response from rest initial conditions to the input $\sin(2t)$? [10]

Properties of the Laplace transform

0. Definition: $\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt$ for $\operatorname{Re} s \gg 0$.

1. Linearity: $\mathcal{L}[af(t) + bg(t)] = aF(s) + bG(s)$.

2. Inverse transform: $F(s)$ essentially determines $f(t)$.

3. s-shift rule: $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

4. t-shift rule: $\mathcal{L}[f_a(t)] = e^{-as}F(s)$, $\quad f_a(t) = u(t-a)f(t-a) = \begin{cases} f(t-a) & \text{if } t > a \\ 0 & \text{if } t < a \end{cases}$.

5. s-derivative rule: $\mathcal{L}[tf(t)] = -F'(s)$.

6. t-derivative rule: $\mathcal{L}[f'(t)] = sF(s) - f(0^-)$, where $f'(t)$ denotes the generalized derivative.

7. Convolution rule: $\mathcal{L}[f(t) * g(t)] = F(s)G(s)$, $f(t) * g(t) = \int_{0^-}^{t^+} f(t-\tau)g(\tau)\,d\tau$.

8. Weight function: $\mathcal{L}[w(t)] = W(s) = 1/p(s)$, $w(t)$ the unit impulse response.

Formulas for the Laplace transform

$$\mathcal{L}[1] = \frac{1}{s} \qquad \mathcal{L}[e^{at}] = \frac{1}{s-a} \qquad \mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$

$$\mathcal{L}[t\cos(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2} \qquad \mathcal{L}[t\sin(\omega t)] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

Fourier coefficients for periodic functions of period 2π:

$$f(t) = \frac{a_0}{2} + a_1\cos(t) + a_2\cos(2t) + \cdots + b_1\sin(t) + b_2\sin(2t) + \cdots$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)\,dt$$

If $\operatorname{sq}(t)$ is the odd function of period 2π which has value 1 between 0 and π, then

$$\operatorname{sq}(t) = \frac{4}{\pi}\left(\sin(t) + \frac{\sin(3t)}{3} + \frac{\sin(5t)}{5} + \cdots\right)$$