18.03SC Practice Problems 26

Convolution

Solution suggestions

Convolution product: The convolution product of two functions $f(t)$ and $g(t)$ is

$$(f * g)(t) = \int_{0^-}^{t^+} f(t-\tau)g(\tau)\,d\tau.$$

This is also a function. We define it only for $t > 0$.

Assertion: Suppose that $w(t)$ is the unit impulse response for the operator $p(D)$. Let $q(t)$ be a (perhaps generalized) function. Then the solution to $p(D)x = q(t)$ with rest initial conditions is given on $t > 0$ by $w(t) * q(t)$.

1. (a) *Compute* $t * 1$. *More generally, compute* $(q * 1)(t)$ *in terms of* $q = q(t)$.

Proceed from the definitions.

$$t * 1 = \int_{0^-}^{t^+} (t-\tau)\cdot 1 d\tau = \int_0^t (t-\tau)d\tau = t\tau - \frac{\tau^2}{2}\Big|_{\tau=0}^{t} = \frac{t^2}{2}.$$

(Here the function we are integrating is continuous at the endpoints of the interval of integration, so we do not need to be careful to write signs on the integral limits.)

In general,

$$q(t) * 1 = \int_{0^-}^{t^+} q(t-\tau)d\tau.$$

So 1 is not a unit for the convolution operation.

(b) *Compute* $1 * t$. *More generally, compute* $(1 * q)(t)$ *in terms of* $q = q(t)$.

Again, proceed from the definitions.

$$1 * t = \int_{0^-}^{t^+} 1\cdot\tau d\tau = \int_0^t \tau d\tau = \frac{\tau^2}{2}\Big|_{\tau=0}^{t} = \frac{t^2}{2}.$$

More generally,

$$(1 * q)(t) = \int_{0^-}^{t^+} 1\cdot q(\tau)d\tau = \int_{0^-}^{t^+} q(\tau)d\tau$$

With the change of variables $t-\tau \to \tau'$ we see that

$$(1 * q)(t) = \int_{0^-}^{t^+} q(\tau)d\tau = \int_{\tau'=t^+}^{\tau'=0^-} q(t-\tau')(-d\tau') = \int_{0^-}^{t^+} q(t-\tau')d\tau' = (q * 1)(t).$$

(Note that here we did have to be careful about keeping track of the signs on the integral limits.)

What we have found here is that $t * 1 = 1 * t$ and $q * 1 = 1 * q$.

In fact, the convolution operation "$*$" is commutative: it can be checked that for any $f(t)$ and $g(t)$, $(f * g)(t) = (g * f)(t)$.

2. *What is the differential operator $p(D)$ whose unit impulse response is the unit step function $u = u(t)$? In **1(b)** you computed $1 * q = u * q$. Is the Assertion in the box in the beginning of this Session true in this case?*

We are looking for the differential operator $P(D)$ such that $P(D)u = \delta$. But we know that $\delta = Du$, so the differential operator we are looking for is just $p(D) = D$.

The Assertion in this case reads that the solution of $Dx = q(t)$ with rest initial conditions is given on $t > 0$ by $x(t) = (u * q)(t) = (1 * q)(t) = \int_{0^-}^{t^+} q(\tau)d\tau$. (Note that convolving with 1 is the same as convolving with $u = u(t)$.)

To check the Assertion we need to show that the derivative of this function over $t > 0$ is $\dot{x}(t) = q(t)$. This is a familiar statement when q is a regular function, but we should be more careful to check that it also holds for generalized inputs. So break up $q(t)$ into the sum of its regular part and its singular part, which is a sum of delta functions. By linearity, it is sufficient to verify the statement for each separately. The statement holds for regular $q(t)$ by the fundamental theorem of calculus. It holds for each $q(t) = \delta_a(t)$ with $a > 0$, since $\int_{0^-}^{t^+} \delta_a(\tau)d\tau$ on $t > 0$ is $u_a(t)$ whose derivative is $\delta_a(t) = q(t)$. It also holds for $q = \delta_0 = \delta$, but that case falls outside the domain over which we are checking the derivative.

Note that to show this we finally had to use the signs on the limits of integration that we had been carefully keeping track of thus far.

3. **(a)** *Assume that $f(t)$ is continuous at $t = a$. What meaning should we give to the product $f(t)\delta(t - a)$?*

This is an important property of the delta function that repeatedly comes up in computation.

Suppose that $d(t)$ is a function representing $\delta(t)$ - i.e., it is nonzero only very near $t = 0$ and has integral 1. Then $f(t)d(t - a)$ is also nonzero only very near $t = a$ and has integral equal to the value of $f(t)$ near a, which, since $f(t)$ is continuous at a, is just $f(a)$. Thus $f(t)d(t - a)$ is a function representing both $f(t)\delta(t - a)$ and $f(a)\delta(t - a)$, and so this product is really a delta function scaled by a constant:

$$f(t)\delta(t - a) = f(a)\delta(t - a).$$

That is, multiplying the continuous function $f(t)$ by $\delta(t - a)$ effectively picks out the value of the function f at $t = a$.

(b) *Assume that $f(t)$ is continuous and $f(t)$ vanishes for $t < 0$. Let a be a positive constant. Explain why $f(t) * \delta(t - a) = f(t - a)$ on $t > 0$.*

With $a = 0$, this shows that $\delta(t)$ serves as a "unit" for the convolution product.

From the definition of convolution,

$$f(t) * \delta(t - a) = \int_{0^-}^{t^+} f(t - \tau)\delta(\tau - a)d\tau.$$

As we just saw in part (a), the product $f(t-\tau)\delta(\tau-a) = f(t-a)\delta(\tau-a)$, so the integral becomes

$$\int_{0^-}^{t^+} f(t-\tau)\delta(\tau-a)d\tau = \int_{0^-}^{t^+} f(t-a)\delta(\tau-a)d\tau = f(t-a)\int_{0^-}^{t^+} \delta(\tau-a)d\tau.$$

For $t \geq a$, the integral in this expression is just 1, and for $0 < t < a$, this integral is zero. So, under our assumption that $f(t)$ is continous and vanishes for $t < 0$, the value of the entire expression over $t > 0$ is the same as $f(t-a)$.

4. (a) *Verify that* $\frac{1}{\omega_n}\sin(\omega_n t)u(t)$ *is the unit impulse response of* $D^2 + \omega_n^2 I$.

Denote $x(t) := \frac{1}{\omega_n}\sin(\omega_n t)u(t)$. Check that $x(t)$ is a homogeneous solution which has $x(t) = 0$ for $t < 0$ and satisfies the required initial conditions to be the unit impulse response of this operator.

Differentiate to obtain $Dx = \cos(\omega_n t)$ and $D^2 x = -\omega_n \sin(\omega_n t)$ (for $t > 0$). So $(D^2 + \omega_n^2 I)x = 0$ when $t > 0$.

Moreover, we can check that x satisfies the required initial conditions in this case: $x(0^+) = 0$ and $\dot{x}(0^+) = 1 = 1/1$, where 1 is the coefficient of the leading term D^2 in this differential operator.

Together this verifies that $\frac{1}{\omega_n}\sin(\omega_n t)u(t)$ is the unit impulse response of $D^2 + \omega_n^2 I$.

(b) *Find the solution to* $\ddot{x} + x = \sin t$ *with initial conditions* $x(0) = \dot{x}(0) = 0$, *using the ERF/resonance.*

We will need to find the general solution to this equation and use the initial conditions to solve for the right constants (i.e. to find the particular solution that satisfies these initial conditions).

As usual, begin by finding a particular solution. The complex replacement of this equation is $\ddot{z} + z = e^{it}$ (and $x = \operatorname{Im} z$). By the Resonant ERF, the complex equation has the particular solution $z_p = \frac{te^{it}}{2i}$. So the original equation has a particular solution $x_p = \operatorname{Im}(z_p) = -\frac{t}{2}\cos t$.

Then read off the homogeneous solutions from the system. This system has characteristic polynomial $r^2 + 1$ with roots $\pm i$, so the homogeneous solutions to this system are of the form $c_1 \cos t + c_2 \sin t$.

So by linearity, the general solution to this equation is

$$x = -\frac{t}{2}\cos t + c_1 \cos t + c_2 \sin t.$$

Now we want $x(0) = c_1 = 0$ and $\dot{x}(0) = -\frac{1}{2} + c_2 = 0$. So we must have $c_1 = 0$ and $c_2 = \frac{1}{2}$, and the particular solution we are looking for is

$$x = -\frac{t}{2}\cos t + \frac{1}{2}\sin t.$$

(c) *By the Assertion,* $\sin t * \sin t$ *should match the solution found in* ***(b)*** *for* $t > 0$. *Verify this by computing* $\sin t * \sin t$ *directly. (Hint:* $\sin(t-\tau) = \sin t \cos\tau - \cos t \sin\tau$.*)*

From the definitions, $\sin t * \sin t$ is

$$\begin{aligned}\int_{0-}^{t+} \sin(t-\tau)\sin\tau d\tau &= \int_0^t \sin t\cos\tau\sin\tau - \cos t\sin^2\tau d\tau \\ &= \int_0^t \sin t\left(\frac{\sin(2\tau)}{2}\right) + \cos t\left(\frac{\cos(2\tau)-1}{2}\right) d\tau,\end{aligned}$$

for $t > 0$. Integrate this out to get

$$\begin{aligned}&\frac{1}{4}[\sin t(-\cos(2\tau)) + \cos t(\sin(2\tau) - 2\tau)]|_{\tau=0}^{\tau=t} \\ &= \frac{1}{4}(-\sin t\cos(2t) + \cos t\sin(2t) - 2t\cos t + \sin t) \\ &= \frac{1}{4}(\sin(2t-t) - 2t\cos t + \sin t) \\ &= \frac{1}{4}(-2t\cos t + 2\sin t) \\ &= -\frac{t}{2}\cos t + \frac{1}{2}\sin t,\end{aligned}$$

for $t > 0$, which, over $t > 0$, matches the solution $x(t)$ we found in **(b).**