18.03SC Practice Problems 26

Convolution

Convolution product: The convolution product of two functions $f(t)$ and $g(t)$ is

$$(f * g)(t) = \int_{0^-}^{t^+} f(t-\tau)g(\tau)\,d\tau.$$

This is also a function. We define it only for $t > 0$.

Assertion: Suppose that $w(t)$ is the unit impulse response for the operator $p(D)$. Let $q(t)$ be a (perhaps generalized) function. Then the solution to $p(D)x = q(t)$ with rest initial conditions is given (on $t > 0$) by $w(t) * q(t)$.

1. (a) Compute $t * 1$. More generally, compute $(q * 1)(t)$ in terms of $q = q(t)$.

(b) Compute $1 * t$. More generally, compute $(1 * q)(t)$ in terms of $q = q(t)$.

Your answers should be related. What general property of the convolution product does this reflect?

2. What is the differential operator $p(D)$ whose unit impulse response is the unit step function $u = u(t)$?

In **1(b)** you computed $1 * q = u * q$. Is the Assertion in the box in the beginning of this Session true in this case?

3. (a) Assume that $f(t)$ is continuous at $t = a$. What meaning should we give to the product $f(t)\delta(t-a)$?

(b) Assume that $f(t)$ is continuous and that $f(t)$ vanishes for $t < 0$. Let a be a nonnegative constant. Explain why $f(t) * \delta(t-a) = f(t-a)$.

With $a = 0$, this shows that $\delta(t)$ serves as a "unit" for the convolution product.

4. (a) Verify that $\frac{1}{\omega_n}\sin(\omega_n t)u(t)$ is the unit impulse response of $D^2 + \omega_n^2 I$.

(b) Find the solution to $\ddot{x} + x = \sin t$ with initial conditions $x(0) = \dot{x}(0) = 0$, using the ERF/resonance.

(c) By the Assertion, $\sin t * \sin t$ should match the solution found in **(b)** for $t > 0$. Verify this by computing $\sin t * \sin t$ directly. (Hint: $\sin(t-\tau) = \sin t\cos\tau - \cos t\sin\tau$.)