Part II Problems and Solutions

Problem 1: *[Convolution]*

(a) *Let $q(t) = \cos(\omega t)$. Compute $w(t) * q(t)$ (where $w(t)$ is the unit impulse response for $D + kI$ and verify that it is the solution to $\dot{x} + kx = q(t)$ with rest initial conditions.*

(b) *Let $q(t) = 1$. Compute $w(t) * q(t)$ (where $w(t)$ is the unit impulse response for $D^2 + \omega_0^2 I$ and verify that it is the solution to $\ddot{x} + \omega_0^2 x = q(t)$ with rest initial conditions.*

(c) *Compute $t^2 * t$ and $t * t^2$. Are they equal?*

(d) *Compute $(t * t) * t$ and $t * (t * t)$. Are they equal?*

Solution: **(a)** We found $w(t)$ in an earlier part I problem: $w(t) = e^{-kt}$.

$$x(t) = w(t) * q(t) = \int_0^t w(t-\tau)q(\tau)\,d\tau = \int_0^t e^{-k(t-\tau)}\cos(\omega\tau)\,d\tau$$

$$= e^{-kt}\int_0^t \mathrm{Re}(e^{(k+i\omega)\tau})\,d\tau = e^{-kt}\mathrm{Re}\frac{e^{(k+i\omega)t}-1}{k+i\omega} =$$

$$\frac{1}{k^2+\omega^2}\mathrm{Re}((k-i\omega)((\cos(\omega t) - e^{-kt}) + i\sin(\omega t)) = \frac{1}{k^2+\omega^2}(k\cos(\omega t) + \omega\sin(\omega t) - ke^{-kt}).$$

Then $\dot{x} = \frac{1}{k^2+\omega^2}(-k\omega\sin(\omega t) + \omega^2\cos(\omega t) + k^2 e^{-kt})$, and indeed $\dot{x} + kx = \cos(\omega t)$. Also, $x(0) = 0$: the convolution chose the transient just right.

(b) We found $w(t)$ in an earlier part I problem: $w(t) = \frac{1}{\omega_0}\sin(\omega_0 t)$.

$$x(t) = w(t) * q(t) = \int_0^t w(t-\tau)q(\tau)\,d\tau = \frac{1}{\omega_0}\int_0^t \sin(\omega_0(t-\tau))\,d\tau$$

$= \frac{1}{\omega_0^2}\cos(\omega_0(t-\tau))\Big|_0^t = \frac{1}{\omega_0^2}(1-\cos(\omega_0 t))$. Then $\dot{x} = \frac{1}{\omega_0}\sin(\omega_0 t)$ and $\ddot{x} = \cos(\omega_0 t)$, so it is true that $\ddot{x} + \omega_0^2 x = 1$. Also $x(0) = 0$ and $\dot{x}(0) = 0$: so rest initial conditions. Once again the convolution integral has chosen just the right homogeneous solution to produce rest initial conditions.

(c) $t^2 * t = \int_0^t (t-\tau)^2\tau\,d\tau = \int_0^t (t^2\tau - 2t\tau^2 + \tau^3)\,d\tau = \frac{1}{2}t^4 - \frac{2}{3}t^4 + \frac{1}{4}t^4 = \frac{1}{12}t^4.$

$t * t^2 = \int_0^t (t-\tau)\tau^2\,d\tau = \int_0^t (t\tau^2 - \tau^3)\,d\tau = \frac{1}{3}t^4 - \frac{1}{4}t^4 = \frac{1}{12}t^4.$

(d) $t * t = \int_0^t (t-\tau)\tau\,d\tau = \int_0^t (t\tau - \tau^2)\,d\tau = \frac{1}{2}t^3 - \frac{1}{3}t^3 = \frac{1}{6}t^3$. Now

$$(t*t)*t = \frac{1}{6}\int_0^t (t-\tau)^3\tau\,d\tau = \frac{1}{6}\int_0^t (t^3 - 3t^2\tau + 3t\tau^2 - \tau^3)\tau\,d\tau = \frac{1}{6}(\frac{1}{2} - \frac{3}{3} + \frac{3}{4} - \frac{1}{5})t^5$$

$= \frac{1}{120}t^5$, while $t * (t*t) = \frac{1}{6}\int_0^t (t-\tau)\tau^3\,d\tau = \frac{1}{6}(\frac{1}{4} - \frac{1}{5})t^5 = \frac{1}{120}t^5.$