# Polynomial Input: The Method of Undetermined Coefficients

## 1. The Basic Result

A *polynomial* is a function of the form

$$q(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0.$$

The largest $k$ for which $a_k \neq 0$ is the *degree* of $q(x)$. (The zero function is a polynomial too, but it doesn't have a degree.)

Note that $q(0) = a_0$ and $q'(0) = a_1$.

Here is the basic fact about the response of an LTI system with characteristic polynomial $p(s)$ to polynomial signals:

**Theorem. (Undetermined coefficients)** If $p(0) \neq 0$, and $q(x)$ is a polynomial of degree $n$, then

$$p(D)y = q(x)$$

has exactly one solution which is a polynomial, and it is of degree $n$.

The best way to see this, and to see how to compute this polynomial particular solution, is by examples.

## 2. The Method of Undetermined Coefficients

Given the linear time invariant (LTI) DE $p(D)y = q(x)$ with $q(x)$ is a polynomial of degree $n$, the Undetermined Coefficient (UC) solution method, as we discussed in the previous note, is to assume a particular solution of the form $y_p = h(x)$, where $h(x)$ is a polynomial of degree $n$ with unknown ("undetermined") coefficients, and then to find the coefficients by substituting $y_p$ into the ODE. It's important to do the work systematically; we suggest following the format given in the following example.

**Example 1.** Find a particular solution $y_p$ to $y'' + 3y' + 4y = 4x^2 - 2x$.

**Solution.** Our trial solution is $y_p = Ax^2 + Bx + C$; we format the work as follows. The lines show the successive derivatives; multiply each line by the factor given in the ODE, and add the equations, collecting like powers of $x$ as you go. The fourth line shows the result; the sum on the left takes into account that $y_p$ is supposed to be a particular solution to the given

ODE.

$$\begin{aligned}
\times 4 \quad y_p &= Ax^2 + Bx + C \\
\times 3 \quad y_p' &= \qquad 2Ax + B \\
y_p'' &= \qquad\qquad 2A \\
4x^2 - 2x &= (4A)x^2 + (4B + 6A)x + (4C + 3B + 2A).
\end{aligned}$$

Equating like powers of $x$ in the last line gives the three equations

$$4A = 4, \qquad 4B + 6A = -2, \qquad 4C + 3B + 2A = 0;$$

solving them in order gives $A = 1$, $B = -2$, $C = 1$, so that $y_p = x^2 - 2x + 1$.

**Example 2.** Solve $y'' + 5y' + 4y = 2x + 3$.

**Solution.** Guess a *trial solution* of the form $y_p = Ax + B$ (same degree as input).

Substitute in DE: $y_p'' + 5y_p' + 4y_p = 0 + 5(A) + 4(Ax + B) = 2x + 3$.

$\Rightarrow\ 4Ax + (5A + 4B) = 2x + 3$.

Equate coefficients: $4A = 2$, $5A + 4B = 3$.

Triangular system is easy to solve: $A = 1/2$, $B = 1/8$.

$\Rightarrow\ y_p = \frac{1}{2}x + \frac{1}{8}$.

Find solution of homogeneous DE: $y'' + 5y' + 4y = 0$

Char. equation: $r^2 + 5r + 4 = 0 \ \Rightarrow\ r = -1, -4$

$\Rightarrow\ y_h = c_1 e^{-t} + c_2 e^{-4t}$

$\Rightarrow$ general solution to DE = $y = y_p + y_h$.

**Example 3.** Solve $y'' + 5y' + 4y = x^2 + 3x$

**Solution.** Guess a **trial solution** of the form $y_p = Ax^2 + Bx + C$ (same degree as input). Substitute this into the DE:

$$y_p'' + 5y_p' + 4y_p = 2A + 5(2Ax + B) + 4(Ax^2 + Bx + C) = x^2 + 3x$$

$\Rightarrow\ 4Ax^2 + (10A + 4B)x + (2A + 5B + 4C) = x^2 + 3x$

Equate coefficients: $4A = 1$, $10A + 4B = 3$, $2A + 5B + 4C = 0$

Triangular system is easy to solve: $A = 1/4$, $B = 1/8$, $C = -9/32$

$\Rightarrow\ y_p = \frac{1}{4}x^2 + \frac{1}{8}x - \frac{9}{32}$.

Use homogeneous solution from previous example to get the general solution to DE: $y = y_p + y_h$.