Undamped Forced Systems

We now look at the pure resonant case for a second-order LTI DE. We will use the language of spring-mass systems in order to interpret the results in physical terms, but in fact the mathematics is the same for any second-order LTI DE for which the coefficient of the first derivative is equal to zero.

The problem is thus to find a particular solution the DE

$$x'' + \omega_0^2 x = F_0 \cos \omega t.$$

The steps, as in the example in the last note, are
Complex replacement: $z'' + \omega_0^2 z \; = \; F_0 e^{i\omega t}, \;\; x = \text{Re}(z)$.
Characteristic polynonial: $p(r) = r^2 + \omega_0^2 \; \Rightarrow \; p(i\omega) = \omega_0^2 - \omega^2$.

$$\text{Exponential Response formula} \Rightarrow \; z_p = \begin{cases} \dfrac{F_0 e^{i\omega t}}{p(i\omega)} & = & \dfrac{F_0 e^{i\omega t}}{\omega_0^2 - \omega^2} & \text{if } w \neq \omega_0 \\ \dfrac{F_0 \, t e^{i\omega t}}{p'(i\omega)} & = & \dfrac{F_0 \, t e^{i\omega t}}{2i\omega} & \text{if } \omega = \omega_0. \end{cases}$$

$$\Rightarrow \; x_p = \begin{cases} \dfrac{F_0 \cos \omega t}{\omega_0^2 - \omega^2} & \text{if } \omega \neq \omega_0 \\ \dfrac{F_0 \, t \sin \omega_0 t}{2\omega_0} & \text{if } \omega = \omega_0 \quad \text{(resonant case)}. \end{cases}$$

Resonance and amplitude response of the undamped harmonic oscillator

In x_p the amplitude = $A = A(\omega) = |\dfrac{F_0}{\omega_0^2 - \omega^2}|$ is a function of ω.

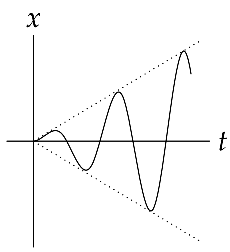
The right plot below shows A as a function of ω. Note, it is similar to the damped amplitude response except the peak is infinitely high. As w gets closer to ω_0 the amplitude increases.

When $\omega = \omega_0$ we have $x_p = \dfrac{F_0 \, t \sin \omega_0 t}{2\omega_0}$. This is called *pure resonance* (like a swing). The frequency ω_0 is called the *resonant* or *natural* frequency of the system.
In the left plot below notice that the response is oscillatory but not periodic. The amplitude keeps growing in time (caused by the factor of t in x_p).

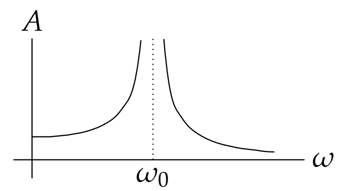
Note carefully the different units and different meanings in the plots below.

The left plot is output vs. time (for a fixed input frequency) and the right plot is output amplitude vs. input frequency.

x and A are in physical units dependent on the system; t is in time; ω is in radians.



Resonant response ($\omega = \omega_0$)



Undamped amplitude response

18.03SC Differential Equations
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