Modes and Roots

A solution of the form $x(t) = ce^{rt}$ to the homogeneous constant coefficient linear equation

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 \dot{x} + a_0 x = 0 \qquad (1)$$

is called a **modal** solution and ce^{rt} is called a **mode** of the system. We saw previously that e^{rt} is a solution exactly when r is a root of the characteristic polynomial

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0.$$

Warning: This only works for *homogeneous constant coefficient* linear equations. It does not work for non-constant coefficient or inhomogeneous or nonlinear equations.

The roots of polynomials can be real or non-real complex numbers. (We need to be a little careful with our language because a real number is also a complex number with imaginary part 0.) Roots can also be repeated. Studying the second order equation will be enough to help us understand all of these possibilities. So, we study (with $a_2 = m$, $a_1 = b$, $a_0 = k$)

$$m\ddot{x} + b\dot{x} + kx = 0. \qquad (2)$$

which models a spring-mass-dashpot system with no external force. The *characteristic equation* is

$$ms^2 + bs + k = 0.$$

1. Real Roots

We have already done this case earlier in this session. If the characteristic polynomial has real roots r_1 and r_2 then the *modal* solutions to (2) are $x_1(t) = e^{r_1 t}$ and $x_2(t) = e^{r_2 t}$. The general solution is found by superposition

$$x(t) = c_1 x_1(t) + c_2 x_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Example 1. (Real roots) Solve the $\ddot{x} + 5\dot{x} + 4x = 0$.

Solution. The characteristic equation is $s^2 + 5s + 4 = 0$. This factors as $(s+1)(s+4) = 0$, so it has roots -1, -4. The modal solutions are $x_1(t) = e^{-t}$ and $x_2(t) = e^{-4t}$. Therefore, the general solution is

$$x(t) = c_1 e^{-t} + c_2 e^{-4t}.$$

2. Complex Roots

(Again, if we were being completely precise, this section would be called *non-real complex roots* to indicate a complex number with non-zero imaginary part.)

Example 2. Solve the equation $\ddot{x} + 4\dot{x} + 5x = 0$.

Solution. The characteristic polynomial is $s^2 + 4s + 5$. Using the quadratic formula the roots are

$$s = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm \sqrt{-1} = -2 \pm i.$$

So our exponential solutions are

$$z_1(t) = e^{(-2+i)t} \quad \text{and} \quad z_2(t) = e^{(-2-i)t}.$$

We use the letter z here to indicate the functions are complex valued.

The general solution is a linear combination of these two *basic* solutions. But, because the DE has real coefficients, we were expecting *real* valued solutions. We will finish this example and get our real solutions after stating and proving the following theorem.

Theorem (Real Solution Theorem):
If $z(t)$ is a complex-valued solution to $m\ddot{z} + b\dot{z} + kz = 0$, where m, b, and k are real, then the real and imaginary parts of z are also solutions.

Proof: Let $u(t)$ be the real part of z and $v(t)$ the imaginary part, so $z(t) = u(t) + iv(t)$. Now, build the table.

$$\begin{array}{rcccccc} k] & & z & = & u & + & iv \\ b] & & \dot{z} & = & \dot{u} & + & i\dot{v} \\ m] & & \ddot{z} & = & \ddot{u} & + & i\ddot{v} \end{array}$$

Summing with the coefficients (and remembering z is a solution to the homogeneous DE) gives

$$(m\ddot{u} + b\dot{u} + ku) + i(m\ddot{v} + b\dot{v} + kv) = 0.$$

Both expressions in parentheses are real, so the only way the sum can be zero is for both of them to be zero. That is, both u and v are solutions of (2) as claimed.

Back to the example: Using Euler's formula

$$z_1(t) = e^{(-2+i)t} = e^{-2t}\cos t + ie^{-2t}\sin t.$$

The real part of $e^{-2t}\cos t + ie^{-2t}\sin t$ is $e^{-2t}\cos t$ and the imaginary part is $e^{-2t}\sin t$. We now have two *basic* solutions and can use superposition to find the general *real valued* solution

$$x(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t).$$

Or we could have also written it as

$$x(t) = e^{-2t}(c_1\cos t + c_2\sin t) = Ae^{-2t}\cos(t-\phi).$$

This is a **damped sinusoid** with **circular pseudo-frequency** 1.

If we had chosen the other exponential solution

$$z_2(t) = e^{(-2-i)t} = e^{-2t}(\cos(-t) + i\sin(-t))$$

then the *basic* real solutions would be

$$e^{-2t}\cos(-t) = e^{-2t}\cos(t) \text{ and } e^{-2t}\sin(-t) = -e^{-2t}\sin(t).$$

Up to a sign these are the same basic solutions as was obtained from z_1, so $z_2(t)$ would have work just as well.

Example 3. Solve $\ddot{x} + \dot{x} + x = 0$.

Solution. Characteristic equation: $s^2 + s + 1 = 0$.
Roots: $\dfrac{-1 \pm \sqrt{1-4}}{2} = \dfrac{-1}{2} \pm i\dfrac{\sqrt{3}}{2}$.
Complex exponential solutions: $z_1(t) = e^{(-1+i\sqrt{3})t/2}$, $z_2(t) = e^{(-1-i\sqrt{3})t/2}$
Basic real solutions: $x_1(t) = \text{Re}(z_1(t)) = e^{-t/2}\cos(\sqrt{3}t/2)$, $\text{Im}(z_1(t)) = e^{-t/2}\sin(\sqrt{3}t/2)$.
General real solution:

$$x(t) = e^{-t/2}(c_1\cos(\sqrt{3}t/2) + c_2\sin(\sqrt{3}\,t/2)) = Ae^{-t/2}\cos(\sqrt{3}\,t/2 - \phi).$$

Example 4. Suppose that the equation $m\ddot{x} + b\dot{x} + kx = 0$ has characteristic roots $a \pm ib$. Give the general real solution.

Solution. In the previous examples we have established a pattern: Two basic real solutions are

$$e^{at}\cos(bt) \quad \text{and} \quad e^{at}\sin(bt)$$

and the general real solution is

$$x(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt) = Ae^{at}\cos(bt-\phi).$$

In words, the real part of the root is the coefficient of t in the exponential and the imaginary part is the angular pseudo-frequency in the trig functions.

For completeness we will walk through the derivation of this. One exponential solution is

$$z_1(t) = e^{(a+ib)t} = e^{at}(\cos(bt) + i\sin(bt)).$$

The two basic solutions are the real and imaginary parts of z_1. That is,

$$e^{at}\cos(bt) \quad \text{and} \quad e^{at}\sin(bt),$$

as claimed.

Example 5. Use the characteristic equation to solve $\ddot{x} + 4x = 0$.

Solution. You should have memorized the solution to this equation. We will check the characteristic equation technique against this known solution.
Characteristic equation: $s^2 + 4 = 0$.
Roots: $s^2 = -4 \Rightarrow s = \pm 2i$.
Complex exponential solutions: $z_1 = e^{2it}$, $z_2 = e^{-2it}$.
Basic real solutions: $\text{Re}(z_1) = \cos(2t)$, $\text{Im}(z_1) = \sin(2t)$.
General real solution:

$$x = c_1\cos(2t) + c_2\sin(2t) = A\cos(2t - \phi)$$

(as expected).

3. Repeated Roots

Example 6. Solve $\ddot{x} + 4\dot{x} + 4x = 0$. Then $p(s) = (s+2)^2$ has r = -2 as a repeated root. The only exponential solution is e^{-2t}. Another solution, which is not a constant multiple of e^{-2t}, is given by te^{-2t}. We will not check this for now, you know how to do it: plug in and use the product rule.

So the general solution is

$$x(t) = c_1e^{-2t} + c_2te^{-2t} \text{ or } x(t) = e^{-2t}(c_1 + c_2t).$$

Example 7. (It's all about the roots)
Suppose the roots –with multiplicity– of a certain homogeneous constant coefficient linear equation are

$$3,\ 4,\ 4,\ 4,\ 5 \pm 2i,\ 5 \pm 2i.$$

Give the general real solution to the equation. What is the order of the equation?

Solution. The basic solutions are

$$e^{3t},\ e^{4t},\ te^{4t},\ t^2e^{4t},\ e^{5t}\cos(2t),\ e^{5t}\sin(2t),\ te^{5t}\cos(2t),\ te^{5t}\sin(2t).$$

(For each repeated root we added a multiple of t to the basic solution.) Using superposition, the general solution is

$$\begin{aligned} x(t) = {} & c_1e^{3t} + c_2e^{4t} + c_3te^{4t} + c_4t^2e^{4t} \\ & + c_5e^{5t}\cos(2t) + c_6e^{5t}\sin(2t) + c_7te^{5t}\cos(2t) + c_8te^{5t}\sin(2t). \end{aligned}$$

There are 8 roots, so the order of the differential equation is 8.