The Characteristic Polynomial

1. The General Second Order Case and the Characteristic Equation

For m, b, k constant, the homogeneous equation

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{1}$$

is a lot like $\dot{x} + kx = 0$, which has as solution $x = e^{-kt}$. We'll be optimistic and try for exponential solutions, $x(t) = e^{rt}$, for some as yet undetermined constant r.

To see which values of r might work, plug $x(t) = e^{rt}$ into (1). Organize the calculation: the $k]$, $b]$, $m]$ are flags indicating that we should multiply the corresponding line by this number.

$$\begin{array}{rrcl} k] & x & = & e^{rt} \\ b] & \dot{x} & = & re^{rt} \\ m] & \ddot{x} & = & r^2e^{rt} \end{array}$$

$$\Rightarrow\ m\ddot{x} + b\dot{x} + kx \ = \ (mr^2 + br + k)e^{rt} \ = \ 0.$$

An exponential is never zero, so we can divide this equation by e^{rt}. We have found that e^{rt} is a solution to (1) exactly when r satisfies the **characteristic equation**

$$mr^2 + br + k = 0.$$

The left hand side is a polynomial called, naturally enough, the **characteristic polynomial** and usually denoted $p(r)$. (You will often also see s used as the variable instead of r. With this notation the characteristic polynomial is $p(s) = ms^2 + bs + k$.)

Example. Find all the solutions to $\ddot{x} + 8\dot{x} + 7x = 0$.

Solution. The characteristic polynomial is $r^2 + 8r + 7$. We want the roots. One reason we wrote out the polynomial was to remind you that you can find roots by factoring it. This one factors as $(r+1)(r+7)$ so the roots are $r = -1$ and $r = -7$, with corresponding exponential solutions are $x_1(t) = e^{-t}$ and $x_2(t) = e^{-7t}$.

By superposition, the *linear combination* of independent solutions gives the general solution:

$$x(t) = c_1e^{-t} + c_2e^{-7t}.$$

Suppose that we have initial conditions $x(0) = 2$ and $\dot{x}(0) = -8$ then we can solve for c_1 and c_2. Use $\dot{x}(t) = -c_1 e^{-t} - 7c_2 e^{-7t}$ and substitute $t = 0$ to get

$$\begin{array}{rcrcrcr} x(0) & = & c_1 & + & c_2 & = & 2 \\ \dot{x}(0) & = & -c_1 & - & 7c_2 & = & -8 \end{array}$$

Adding these two equations yields $-6c_2 = -6$, so $c_2 = 1$ and $c_1 = 1$. The solution to our DE with the given initial conditions is then $x(0) = 2$, $\dot{x}(0) = -8$ is

$$x(t) = e^{-t} + e^{-7t}.$$

2. The General *n*th Order Case

In the same way we can take the homogeneous constant coefficient linear equation of degree n

$$a_n x^{(n)} + \cdots + a_1 \dot{x} + a_0 x = 0$$

and get its *characteristic polynomial,*

$$p(r) = a_n r^n + \cdots + a_1 r + a_0$$

The exponential $x(t) = e^{rt}$ is a solution of the homogeneous DE if and only if r is a root of $p(r)$, i.e. $p(r) = 0$. By superposition, any linear combination of these exponentials is also a solution.