

## Operator Rules

Our work with these differential operators will be based on several rules they satisfy. In stating these rules, we will always assume that the functions involved are sufficiently differentiable, so that the operators can be applied to them.

**Sum rule.** If  $p(D)$  and  $q(D)$  are polynomial operators, then for any (sufficiently differentiable) function  $u$ ,

$$[p(D) + q(D)]u = p(D)u + q(D)u . \quad (1)$$

**Linearity rule.** If  $f$  and  $g$  are functions and  $c_1$  and  $c_2$  are constants,

$$p(D)(c_1f + c_2g) = c_1p(D)f + c_2p(D)g . \quad (2)$$

**Proof of the linearity rule:** This rule follows from the linearity of differentiation. That is,

$$D(c_1f + c_2g) = (c_1f + c_2g)' = c_1f' + c_2g' = c_1Du_1 + c_2Du_2.$$

Similarly taking the second or higher derivative also follows the linearity rule . That is,

$$D^n(c_1f + c_2g) = \frac{d^n}{dt}(c_1f + c_2g) = c_1f^{(n)} + c_2g^{(n)} = c_1D^n f + c_2D^n g.$$

Next, we can scale the linear operator  $D^n$  by  $a$  and it stays linear. That is,

$$aD^n(c_1f + c_2g) = a \frac{d^n}{dt}(c_1f + c_2g) = c_1af^{(n)} + c_2ag^{(n)} = c_1aD^n f + c_2aD^n g$$

(Notice that  $a$  does not actually have to be a constant, it can be a function of  $t$  (or of whatever independent variable we're using). )

Finally we can combine these operators into a polynomial operator

$$D^n + a_1D^{n-1} + \dots + a_{n-1}D + a_n$$

which clearly still obeys the linearity rule. □

**Multiplication rule.** If  $p(D) = g(D)h(D)$  as polynomials in  $D$ , then

$$p(D)u = g(D)(h(D)u). \quad (3)$$

The picture illustrates the meaning of the right side of (3). The property is true when  $h(D)$  is the simple operator  $aD^k$ , essentially because

$$D^m(aD^k u) = aD^{m+k}u.$$

It extends to general polynomial operators  $h(D)$  by linearity. Note that here  $a$  must be a *constant*; it's false otherwise.

An important corollary of the multiplication property is that *polynomial operators with constant coefficients commute*; i.e., for every function  $u(t)$ ,

$$g(D)(h(D)u) = h(D)(g(D)u). \quad (4)$$

As polynomials,  $g(D)h(D) = h(D)g(D) = p(D)$  therefore by the multiplication rule, both sides of (4) are equal to  $p(D)u$  and therefore equal to each other.

The remaining two rules are of a different type and are more concrete: they tell us how polynomial operators behave when applied to exponential functions and products involving exponential functions.

**Substitution rule.**

$$p(D)e^{at} = p(a)e^{at} \quad (5)$$

**Proof.** We have, by repeated differentiation,

$$De^{at} = ae^{at}, \quad D^2e^{at} = a^2e^{at}, \quad \dots, \quad D^ke^{at} = a^ke^{at};$$

therefore,

$$(D^n + c_1D^{n-1} + \dots + c_n)e^{at} = (a^n + c_1a^{n-1} + \dots + c_n)e^{at},$$

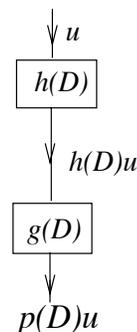
which is the substitution rule (5). □

**The exponential-shift rule** This handles expressions such as  $t^k e^{at}$  and  $t^k \sin at$ . Let  $u = u(t)$ . Then

$$p(D)e^{at}u = e^{at}p(D+a)u. \quad (6)$$

**Proof.** We prove it in successive stages. First, it is true when  $p(D) = D$ , since by the product rule for differentiation,

$$De^{at}u(t) = e^{at}Du(t) + ae^{at}u(t) = e^{at}(D+a)u(t). \quad (7)$$



To show the rule is true for  $D^k$ , we apply (7) to  $D$  repeatedly:

$$\begin{aligned} D^2 e^{at} u &= D(D e^{at} u) &= D(e^{at}(D+a)u) && \text{by (7);} \\ & &= e^{at}(D+a)((D+a)u), && \text{by (7);} \\ & &= e^{at}(D+a)^2 u, && \text{by (3).} \end{aligned}$$

In the same way,

$$\begin{aligned} D^3 e^{at} u &= D(D^2 e^{at} u) &= D(e^{at}(D+a)^2 u) && \text{by the above;} \\ & &= e^{at}(D+a)((D+a)^2 u), && \text{by (7);} \\ & &= e^{at}(D+a)^3 u, && \text{by (3),} \end{aligned}$$

and so on. This shows that (6) is true for an operator of the form  $D^k$ . To show it is true for a general operator

$$p(D) = D^n + a_1 D^{n-1} + \dots + a_n,$$

we write (6) for each  $D^k(e^{at}u)$ , multiply both sides by the coefficient  $a_k$ , and add up the resulting equations for the different values of  $k$ .  $\square$

MIT OpenCourseWare  
<http://ocw.mit.edu>

18.03SC Differential Equations  
Fall 2011

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.