

p(D) Notation

We start by recalling the basic results we have developed so far. We are studying solutions $x = x(t)$ of the linear constant coefficient DE

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_1 x' + a_0 x = q(t) \quad (\text{I})$$

with characteristic polynomial

$$p(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 \quad (\text{P})$$

and homogeneous case ($q = 0$)

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \dots + a_1 x' + a_0 x = 0 \quad (\text{H})$$

Notice that the left-hand sides of (I) and (H) have the same form. For this reason it will be useful to have a more compact notation. This is in fact provided by an important mathematical tool called **operators**. We will study these in more detail in the session on linear operators. For now, we just note that we can write $D = \frac{d}{dt}$ for the operation of differentiation applied to functions of t , i.e. if $x = x(t)$, then $Dx = \frac{dx}{dt}$, the first derivative of x . In the same way we can write $D^2 = \frac{d^2}{dt^2}$ for differentiation twice, i.e. $D^2 x = \frac{d^2 x}{dt^2}$, the second derivative of $x = x(t)$; similarly $D^3 = \frac{d^3}{dt^3}$ for differentiation three times, and so on. Then if $p(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0$ is any polynomial, we can write

$$p(D) = a_n D^n + a_{n-1} D^{n-1} + \dots + a_1 D + a_0.$$

The DE's (I) and (H) then become the statements

$$p(D)x = q \quad (\text{I})$$

$$p(D)x = 0 \quad (\text{H})$$

respectively – an efficient way to write the DE's indeed!

Now let's recall the basics, but with our new operator notation. For the homogeneous case we have the following key theorem.

Transience theorem. All solutions $x = x(t)$ to the linear homogeneous constant coefficient DE

$$p(D)x = 0 \quad (\text{H})$$

decay to zero as $t \rightarrow \infty$ exactly when all roots r of the characteristic polynomial $p(s)$ have negative real part.

In this case the solutions to (H) are called **transients**. By superposition, all solutions to (I) then converge to the *same* solution as t gets large, and we say that the DE is **stable**.

If we have a system modeled by a stable equation, but we are only interested in what it looks like after the transients have died down, we can ignore the initial condition:



So in this case we are looking for particular solutions x_p . If the input signal is sinusoidal, then we know from the results we obtained in the last session that there will be a particular solution which is also sinusoidal. This is the unique steady state solution which is *periodic* and it is of particular importance in many applications. Let's review how it goes and then introduce some useful definitions and terminology that apply to these solutions.

The starting point is the Exponential Response Formula (ERF), which in the operator notation reads

$$p(D)x = Be^{at}$$

and has a solution

$$x_p = \frac{Be^{at}}{p(a)}$$

provided $p(a) \neq 0$.

As we saw in the last session, the ERF and complex replacement can be used to obtain the periodic solution x_p to the DE with sinusoidal input, i.e.

$$p(D)x = B \cos(\omega t).$$

This is done as follows. Since $B \cos(\omega t) = B \operatorname{Re}(e^{i\omega t})$ we look at the complex equation

$$p(D)z = Be^{i\omega t}, \quad \text{so } x = \operatorname{Re}(z).$$

The exponential response formula gives

$$z_p = \frac{B}{p(i\omega)} e^{i\omega t} \Rightarrow x_p = B \operatorname{Re} \left(\frac{e^{i\omega t}}{p(i\omega)} \right)$$

Thus,

$$x_p = \frac{B}{|p(i\omega)|} \cos(\omega t - \phi),$$

where $\phi = \text{Arg}(p(i\omega))$. The solution x_p is the particular steady-state periodic solution.

Let's examine the relation between the periodic input $q(t) = B \cos(\omega t)$ and its periodic output $x_p(t) = \frac{B}{|p(i\omega)|} \cos(\omega t - \phi)$. We see that the amplitude of the input B is scaled and becomes the amplitude $\frac{B}{|p(i\omega)|}$ of the output. We also see that the output sinusoid $x_p(t)$ is shifted by an angle $\phi = \text{Arg}(p(i\omega))$ relative to the input sinusoid $q(t)$.

This motivates the following definitions: for a CC linear DE

$$P(D)x = q(t) \quad \text{with sinusoidal input } q(t).$$

Definition:

1. The **gain** is defined to be the *ratio of the amplitude of the output sinusoid to the amplitude of the input sinusoid*.
2. The **phase lag** is defined to be the *angle by which the output sinusoid is shifted relative to the input sinusoid*.

In the special case $q(t) = B \cos \omega t$ which we solved above, we have that the gain g and the phase lag ϕ are

$$g = \frac{1}{|p(i\omega)|}, \quad \phi = \text{Arg}(p(i\omega)).$$

When solving using $p(D)x = B \cos \omega t$ by complex replacement and the ERF we have $x_p = \text{Re}(z_p)$ where $z_p(t)$ is the complex solution to $p(D)z_p = B e^{i\omega t}$. That is

$$z_p = \frac{B}{p(i\omega)} e^{i\omega t}.$$

For this reason, we define the **complex gain** in this case as $\frac{1}{p(i\omega)}$.

Note that the gain and the phase lag depend only on the frequency of the ω of the input signal (as well as on the system $p(D)$ of course).

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