Mechanical Vibration System: Driving Through the Spring

The figure below shows a spring-mass-dashpot system that is driven through the spring.

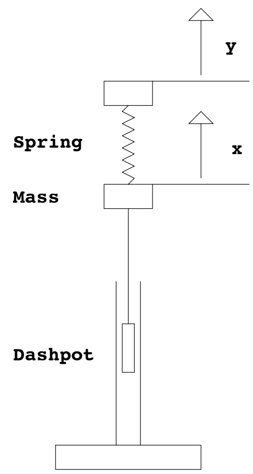

Figure 1. Spring-driven system

Suppose that y denotes the displacement of the plunger at the top of the spring and $x(t)$ denotes the position of the mass, arranged so that $x = y$ when the spring is unstretched and uncompressed. There are two forces acting on the mass: the spring exerts a force given by $k(y - x)$ (where k is the spring constant) and the dashpot exerts a force given by $-b\dot{x}$ (against the motion of the mass, with damping coefficient b). Newton's law gives

$$m\ddot{x} = k(y - x) - b\dot{x}$$

or, putting the system on the left and the driving term on the right,

$$m\ddot{x} + b\dot{x} + kx = ky\,. \qquad (1)$$

In this example it is natural to regard y, rather than the right-hand side $q = ky$, as the input signal and the mass position x as the system response. Suppose that y is sinusoidal, that is,

$$y = B_1 \cos(\omega t).$$

Then we expect a sinusoidal solution of the form

$$x_p = A\cos(\omega t - \phi).$$

By definition the *gain* is the ratio of the amplitude of the system response to that of the input signal. Since B_1 is the amplitude of the input we have $g = A/B_1$.

In the previous note in this session, we worked out the formulas for g and ϕ, and so we can now use them with the following small change. The k on the right-hand-side of equation (1) needs to be included in the gain (since we don't include it as part of the input). We get

$$g(\omega) = \frac{A}{B_1} = \frac{k}{|p(i\omega)|} = \frac{k}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$
$$\phi(\omega) = \tan^{-1}\left(\frac{b\omega}{k - m\omega^2}\right).$$

Note that the gain is a function of ω, i.e. $g = g(\omega)$. Similarly, the *phase lag* $\phi = \phi(\omega)$ is a function of ω. The entire story of the steady state system response $x_p = A\cos(\omega t - \phi)$ to sinusoidal input signals is encoded in these two functions of ω, the gain and the phase lag.

We see that choosing the input to be y instead of ky scales the gain by k and does not affect the phase lag.

The factor of k in the gain does not affect the frequency where the gain is greatest, i.e. the practical resonant frequency. From the previous note in this session we know this is

$$\omega_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}.$$

Note: Another system leading to the same equation is a series RLC circuit. We will favor the mechanical system notation, but it is interesting to note the mathematics is exactly the same for both systems.