Frequency Response and Practical Resonance

In the previous note in this session we found the periodic solution to the equation

$$mx'' + bx' + kx = B\cos(\omega t). \tag{1}$$

The solution was $x_p = gB\cos(\omega t - \phi)$, where g is the gain

$$g = g(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}} \tag{2}$$

and ϕ is the phase lag

$$\phi = \phi(\omega) = \text{Arg}(p(i\omega)) = \tan^{-1}(b\omega/(k - m\omega^2)). \tag{3}$$

The gain or amplitude response is a function of ω. It tells us the size of the system's response to the given input frequency. If the amplitude has a peak at ω_r we call this the **practical resonance frequency.** If the damping b gets too large then, for the system in equation (1), there is no peak and, hence, no practical resonance. The following figure shows two graphs of $g(\omega)$, one for small b and one for large b.

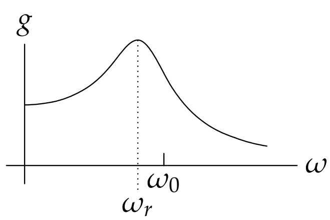


Fig 1a. Small b (has resonance).

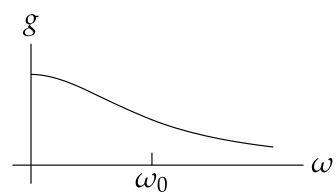


Fig 1b. Large b (no resonance)

In figure (1a) the damping constant b is small and there is practical resonance at the frequency ω_r. In figure (1b) b is large and there is no practical resonant frequency.

Finding the Practical Resonant Frequency.

We now turn our attention to finding a formula for the practical resonant frequency -if it exists- of the system in (1). Practical resonance occurs at the frequency ω_r where $g(w)$ has a maximum. For the system (1) with gain (2) it is clear that the maximum gain occurs when the expression under the radical has a minimum. Accordingly we look for the minimum of

$$f(\omega) = (k - m\omega^2)^2 + b^2\omega^2.$$

Setting $f'(\omega) = 0$ and solving gives

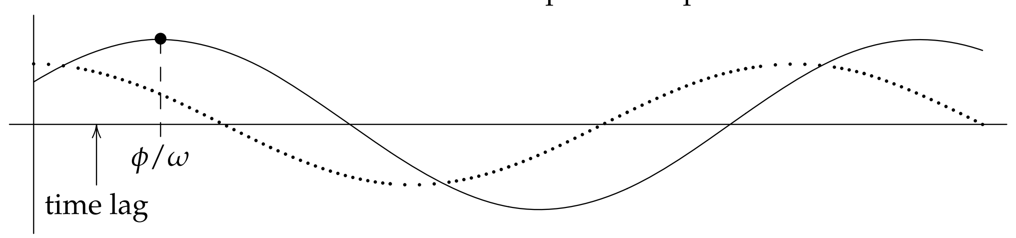
$$f'(\omega) = -4m\omega(k - m\omega^2) + 2b^2\omega = 0$$
$$\Rightarrow \omega = 0 \text{ or } m^2\omega^2 = mk - b^2/2.$$

We see that if $mk - b^2/2 > 0$ then there is a practical resonant frequency

$$\omega_r = \sqrt{\frac{k}{m} - \frac{b^2}{2m^2}}.$$

Phase Lag:
In the picture below the dotted line is the input and the solid line is the response.

The damping causes a lag between when the input reaches its maximum and when the output does. In radians, the angle ϕ is called the *phase lag* and in units of time ϕ/ω is the *time lag*. The lag is important, but in this class we will be more interested in the amplitude response.



18.03SC Differential Equations
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