

18.03SC Practice Problems 14

Frequency response

Solution suggestions

In this problem session we will work with a second order mass-spring-dashpot system driven by a force F_{ext} acting directly on the mass: $m\ddot{x} + b\dot{x} + kx = F_{\text{ext}}$. So here the input signal is F_{ext} and the system response is x . (Contrast this with the mass-spring-dashpot system examined previously in Session 10.)

We will be interested in a sinusoidal input signal, $F_{\text{ext}}(t) = A \cos(\omega t)$, and in the steady state, also sinusoidal, system response, $x_p(t) = gA \cos(\omega t - \phi)$. Here g is the gain of the system and ϕ is the phase lag. Both depend upon ω , and we will examine these dependencies. This setup is illustrated by the Mathlet *Amplitude and Phase: Second Order IV*, which you can have open in parallel to check your intuition as you work through this problem session.

Take $m = 1$, $b = \frac{1}{4}$, and $k = 2$. Also, assume $A = 1$, so that the amplitude of the system response is just the gain. You should be able to select these values on the Mathlet and get an animation of the exact system we will be examining.

1. Write down the equation for the system we are working with. Compute its complex gain $H(\omega)$. (That is, make the complex replacement $F_{\text{cx}} = e^{i\omega t}$ for the input signal, and express the exponential system response z_p as a complex multiple of F_{cx} : $z_p = H(\omega)F_{\text{cx}}$.)

Here the complex replacement of the input signal is $F_{\text{cx}} = e^{i\omega t}$, so the complex replacement of the equation is $\ddot{z} + \frac{1}{4}\dot{z} + 2z = F_{\text{cx}} = e^{i\omega t}$. This system has characteristic polynomial $p(s) = s^2 + \frac{1}{4}s + 2$, and $p(i\omega) = -\omega^2 + \frac{i}{4}\omega + 2 \neq 0$, so we can apply the ERF to solve. By the ERF, $z_p = e^{i\omega t} / p(i\omega) = F_{\text{cx}} / p(i\omega)$. Thus,

$$H(\omega) = \frac{z_p}{F_{\text{cx}}} = \frac{1}{p(i\omega)} = \frac{1}{2 - \omega^2 + i(\omega/4)}.$$

2. Write down the expression for the gain $g(\omega) = |H(\omega)|$. What is the amplitude of the system response when $\omega = 1$? Use the Mathlet to check your answer.

In general, the gain is

$$g(\omega) = |H(\omega)| = \frac{1}{|p(i\omega)|} = \frac{1}{\sqrt{(2 - \omega^2)^2 + (\omega/4)^2}} = \left(\omega^4 - \frac{63}{16}\omega^2 + 4 \right)^{-\frac{1}{2}}.$$

For $A = 1$, the amplitude of the system response is equal to the gain $g(\omega)$. So when $\omega = 1$, the amplitude of the system response is

$$g(1) = \frac{1}{\sqrt{(5 \cdot 16 - 63)/16}} = \frac{4}{\sqrt{17}}.$$

To check this on the Mathlet, select "Bode Plots" and move the slider for ω to $\omega = 1$. You can then estimate the amplitude from the top plot.

3. What is the resonant angular frequency ω_r ? (Hint: minimize the square of the denominator.) Again, you should be able to check your answer on the Mathlet.

The resonant angular frequency ω_r is the angular frequency at which the gain $g(\omega)$ is maximal, if such exists. So, to find ω_r , we want to maximize $g(\omega)$. In this particular case, this is equivalent to minimizing the square of the reciprocal of the gain, $g(\omega)^{-2} = \omega^4 - \frac{63}{16}\omega^2 + 4$.

Begin by finding the critical points of $\omega^4 - \frac{63}{16}\omega^2 + 4$. Take the first derivative in ω and set it to zero to get the equation

$$4\omega^3 - \frac{63}{8}\omega = 0,$$

which implies $\omega = 0$ or $\omega^2 = \frac{63}{32}$, i.e., $\omega = \pm \frac{3\sqrt{14}}{8}$. For us only positive values of frequency are meaningful, so the resonant frequency is

$$\omega_r = \frac{3\sqrt{14}}{8} \approx 1.40.$$

(The leading coefficient of the cubic is positive and there are three distinct zeroes, so the right-most one corresponds to a minimum of the original quartic equation for the square of the denominator of the gain, and, therefore, to a maximum of the gain itself.)

You can check on the Mathlet that the graph for A seems to attain its peak at approximately 1.40.

4. On the Mathlet it appears that the phase lag is almost, but not quite, $\frac{\pi}{2}$ at the resonant angular frequency. Verify this computationally: at what frequency is the phase lag exactly one quarter cycle?

From the plot of $-\phi$ on the Mathlet, at $\omega = 1.40$, the negative phase lag $-\phi \approx -85^\circ$, so yes, it appears that the phase lag is close to, but not equal to, $\pi/2$ at the resonant frequency.

The phase lag will be exactly $\pi/2$ when the complex gain H is purely (positive) imaginary. Equivalently, this is when the denominator of H is purely imaginary, i.e., when $2 - \omega^2 = 0$. Taking the positive root of this equation, we see that the phase lag is one quarter cycle at frequency

$$\omega = \sqrt{2}.$$

Again, note that $\sqrt{2} \approx 1.414$ is close to, but not the same as, the resonant frequency of $\frac{3\sqrt{14}}{8} \approx 1.40$.

5. At what angular frequency is the phase lag equal to $\frac{\pi}{4}$? How about $\frac{3\pi}{4}$? Are your answers supported by the Mathlet?

As in question 4, the phase lag ϕ is $\frac{\pi}{4}$ when the real and imaginary parts of $p(i\omega)$ are equal (and positive). This corresponds to the equation $2 - \omega^2 = \omega/4$, which has solutions $\omega = \frac{-1 \pm \sqrt{129}}{8}$. We take the positive value, so

$$\omega_{\phi=\pi/4} = \frac{-1 + \sqrt{129}}{8} \approx 1.29.$$

Similarly, the phase lag is $\frac{3\pi}{4}$ when the real part of $p(i\omega)$ is the negative of the imaginary part. This gives the equation $2 - \omega^2 = -\omega/4$, which has solutions $\omega = \frac{1 \pm \sqrt{129}}{8}$. Again we take the positive value to get

$$\omega_{\phi=3\pi/4} = \frac{1 + \sqrt{129}}{8} \approx 1.54.$$

You can move the marker for ω to 1.30 and 1.55 in the Bode plots on the Mathlet to visually check these values.

6. *New topic: Find a solution of $\ddot{x} + 3\dot{x} + 2x = te^{-t}$.*

We use the method of variation of parameters.

Suppose there is a solution of the form $x(t) = u(t)e^{-t}$ for some function $u(t)$. Then $\dot{x} = \dot{u}e^{-t} - ue^{-t}$ and $\ddot{x} = \ddot{u}e^{-t} - 2\dot{u}e^{-t} + ue^{-t}$, and the equation becomes

$$e^{-t}(\ddot{u} + \dot{u}) = te^{-t}.$$

The exponential is never zero, so can cancel e^{-t} from both sides and obtain

$$\ddot{u} + \dot{u} = t.$$

To solve this equation for u , we use the method of undetermined coefficients. However, the corresponding characteristic polynomial $p(s) = s^2 + s$ has zero constant term and nonzero linear term, this means the term on the right hand side satisfies the homogeneous system given by $p(D)$, but not the one given by $p'(D)$. So we should guess a quadratic polynomial solution, $u = at^2 + bt + c$. Plugging this in to the equation and solving for the coefficients gives us that $u = \frac{t^2}{2} - t$.

So, going back to the original equation, one solution is given by

$$x = \left(\frac{t^2}{2} - t\right)e^{-t}.$$

Note that here checking your answer would take some effort.

7. *Explain why a mass-spring-dashpot system driven through the spring is modeled by the equation $m\ddot{x} + b\dot{x} + kx = ky$. Here x measures the displacement of the mass from its position at rest, y measures the displacement of the other end of the spring from its position at rest by the driving force, and $x = y$ when the spring is relaxed.*

Here is a picture of a mass-spring-dashpot system, driven through the spring.

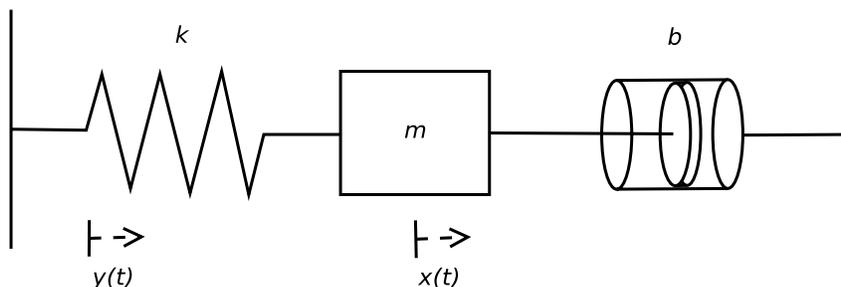


Figure 2: A mass-spring-dashpot system, driven through the spring.

The variable we are solving for, $x = x(t)$, is measuring the displacement of the mass and, therefore, also the displacement of the mass side of the spring (the “right” side on the picture) from some chosen position at rest, while the input $y = y(t)$ is measuring the displacement of the other side of the spring (the “left” side) from some chosen reference point. The phrasing of the problem implies that when both x and y are zero, the spring is relaxed. A choice of reference points and positive directions for x and y is included in the picture.

We set up the differential equation for the displacement of the mass x from the second law of motion

$$\text{mass} \cdot \text{acceleration} = \text{net force}$$

by finding and writing out the net force acting on the mass in terms of the variables given and some constants that we’ll introduce. In this closed system the mass is acted upon by two forces: the spring force F_{spring} and the damping force $F_{dashpot}$.

The spring is stretched by a distance $x(t) - y(t)$ from its relaxed position. (In particular, as indicated, the spring is relaxed when $x(t) = y(t)$.) The spring law states that $F_{spring} = -k \cdot (x(t) - y(t)) = k(y(t) - x(t))$, where k denotes the spring constant.

The velocity of the mass is the time-derivative of displacement, \dot{x} . So the dashpot law states that there is a damping force acting on the mass that is proportional to the velocity of the mass, i.e. $F_{dashpot} = -b\dot{x}$ for some damping constant b . (The signs are chosen so that in a real-world system damping would correspond to a positive damping constant.)

The acceleration of the mass is \ddot{x} , and the value of the mass can be denoted by the constant m . Combining all of this, we obtain the equation

$$m\ddot{x} = F_{spring} + F_{dashpot} = k(y(t) - x(t)) - b\dot{x},$$

or, in standard form, $m\ddot{x} + b\dot{x} + kx = ky$, as desired.

This situation is modeled in the Mathlet [Amplitude and Phase: Second Order I](#). This mathlet assumes unit mass and drives the spring by a sinusoidal y . It draws the system vertically, but, just as we do here, does not intend to explicitly include other external forces like gravity in the closed system it is modeling.

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