18.03SC Unit 2 Practice Exam and Solutions

Study guide

1. Models. A linear differential equation is one of the form $a_n(t)x^{(n)} + \cdots + a_1(t)\dot{x} + a_0(t)x = q(t)$. The $a_k(t)$ are "coefficients." The left side models a system, $q(t)$ arises from an input signal, and solutions $x(t)$ provide the system response. In this course the system is unchanging—time-invariant—so the coefficients are constant. Then the equation can be written in terms of the characteristic polynomial $p(s) = a_n s^n + \cdots + a_1 s + a_0$ as $p(D)x = q(t)$.

Spring system: $p(s) = ms^2 + bs + k$. System response x is position of the mass. If driven directly, $q(t) = F_{ext}(t)$. If driven through the spring, $q(t) = ky(t)$ ($y(t)$ the position of the far end of the spring). If driven throught the dashpot, $q(t) = b\dot{y}$ (y=position of far end of dashpot).

2. Homogeneous equations. The "mode" e^{rt} solves $p(D)x = 0$ exactly when $p(r) = 0$. If r is a double root one needs te^{rt} also (etc.). The general solution is a linear combination of these (Super I). If the coefficients are real and $r = a + bi$ with $b \neq 0$ then $e^{at}\cos(bt)$ and $e^{at}\sin(bt)$ are independent real solutions. If all roots have negative real part then all solutions decay to zero as $t \to \infty$ and are called *transients*. In case $p(s) = ms^2 + bs + k$ with $m > 0$ and $b, k \geq 0$, the equation is *overdamped* if the roots are real and distinct ($k < b^2/4m$), *underdamped* if the roots are not real ($k > b^2/4m$), and *critically damped* if there is just one (repeated) root ($k = b^2/4m$). In the underdamped case the general solution is $Ae^{-bt/2m}\cos(\omega_d t - \phi)$ where $\omega_d = \sqrt{\frac{k}{m} - (\frac{b}{2m})^2}$ is the *damped circular frequency*.

3. Linearity. Superposition III: if $p(D)x_1 = q_1(t)$ and $p(D)x_2 = q_2(t)$, then $x = c_1x_1 + c_2x_2$ solves $p(D)x = c_1q_1(t) + c_2q_2(t)$ (c_1, c_2 constant). Consequence (Super II): the general solution to $p(D)x = q(t)$ is $x = x_p + x_h$ where x_p is a solution and x_h is the general solution to $p(D)x = 0$.

4. Exponential response formula: If $p(r) \neq 0$ then $Ae^{rt}/p(r)$ solves $p(D)x = Ae^{rt}$. If $p(r) = 0$ but $p'(r) \neq 0$ then $Ate^{rt}/p'(r)$ solves $p(D)x = Ae^{rt}$. (Etc.)

5. Complex replacement: If $p(s)$ has real coefficients then solutions of $p(D)x = Ae^{rt}\cos(\omega t)$ are real parts of solutions of $p(D)z = Ae^{(r+i\omega)t}$.

6. Undetermined coefficients: With $p(s) = a_n s^n + \cdots + a_1 s + a_0$, if $a_0 \neq 0$ then $p(D)x = b_k t^k + \cdots + b_1 t + b_0$ has exactly one polynomial solution, which has degree at most k. If a_k is the first nonzero coefficient, then make the substitution $u = x^{(k)}$ and proceed ("reduction of order"). For x_p you can take constants of integration to be zero.

7. Variation of parameters: To solve $p(D)x = f(t)e^{rt}$, try $x = ue^{rt}$. This leads to a different equation for u with right hand side $f(t)$.

8. Time invariance: If $p(D)x = q(t)$, then $y = x(t-a)$ solves $p(D)y = q(t-a)$. This lets you convert any sinusoidal term in $q(t)$ to a cosine.

9. Frequency response: An input signal y determines $q(t)$ in $p(D)x = q(t)$. With $y = y_{\text{cx}} = e^{i\omega t}$, an exponential system response has the form $H(\omega)e^{i\omega t}$ for some complex number $H(\omega)$, calculated using ERF. (If ERF fails then the complex gain is infinite.) Then with $y = A\cos(\omega t)$, $x_p = g\cos(\omega t - \phi)$ where $g = |H(\omega)|$ is the *gain* and $\phi = -\text{Arg}\,(H(\omega))$ is the phase lag. By time invariance the gain and phase lag are the same for any sinusoidal input signal of circular frequency ω.

Practice Hour Exam

1. The mass and spring constant in a certain mass/spring/dashpot system are known—$m = 1$, $k = 25$—but the damping constant b is not known. It's observed that for a certain solution $x(t)$ of $\ddot{x} + b\dot{x} + 25x = 0$, $x(\frac{\pi}{6}) = 0$ and $x(\frac{\pi}{2}) = 0$, but $x(t) > 0$ for $\frac{\pi}{6} < t < \frac{\pi}{2}$.

(a) Is the system underdamped, critically damped, or overdamped?

(b) Determine the value of b.

2. Find a solution of $3\ddot{x} + 2\dot{x} + x = t^2$.

3. Find a solution to $\ddot{x} + 3\dot{x} + 2x = e^{-t}$.

4. This problem concerns the sinusoidal solution $x(t)$ of $\ddot{x} + 4\dot{x} + 9x = \cos(\omega t)$.

(a) For what value of ω is the amplitude of $x(t)$ maximal?

(b) For what value of ω is the phase lag exactly $\frac{\pi}{4}$?

5. The equation $2\ddot{x} + \dot{x} + x = \dot{y}$ models a certain system in which the input signal is y and the system response is x. We drive it with a sinusoidal input signal of circular frequency ω. Determine the complex gain as a function of ω, and the gain and phase lag at $\omega = 1$.

6. Find a solution to $\dfrac{d^3x}{dt^3} + x = e^{-t}\cos t$.

7. Assume that $\cos t$ and t are both solutions of the equation $p(D)x = q(t)$, for a certain polynomial $p(s)$ and a certain function $q(t)$.

(a) Write down a nonzero solution of the equation $p(D)x = 0$.

(b) Write down a solution $x(t)$ of $p(D)x = q(t)$ such that $x(0) = 2$.

(c) Write down a solution of the equation $p(D)x = q(t-1)$.

Solutions

1. (a) Underdamped.

(b) The pseudoperiod is $2(\frac{\pi}{2} - \frac{\pi}{6}) = \frac{2\pi}{3}$. Thus $\omega_d = \frac{2\pi}{2\pi/3} = 3$, $9 = \omega_d^2 = k - (b/2)^2 = 25 - (b/2)^2$, so $(b/2)^2 = 25 - 9 = 16$, $b/2 = 4$, $b = 8$.

2.

$$\begin{array}{rrcccccc} 1] & x & = & at^2 & + & bt & + & c \\ 2] & \dot{x} & = & & & 2at & + & b \\ 3] & \ddot{x} & = & & & & & 2a \\ \hline & t^2 & = & at^2 & + & (b+4a)t & + & c+2b+6a \end{array}$$

so $a = 1$, $b + 4a = 0$, $c + 2b + 6a = 0$, $b = -4$, $c = 2$: $x_p = t^2 - 4t + 2$.

3. $p(s) = s^2 + 3s + 2$, $p(-1) = (-1)^2 + 3(-1) + 2 = 0$, so ERF fails. $p'(s) = 2s + 3$, $p'(-1) = 1$, $x_p = te^{-t}$.

4. (a) The amplitude is $1/|p(i\omega)|$. $p(i\omega) = (k - m\omega^2) + bi\omega = (9 - \omega^2) + 4i\omega$. To maximize the amplitude we can minimize $|p(i\omega)|^2 = (9 - \omega^2)^2 + 16\omega^2$. Now $\frac{d}{d\omega}|p(i\omega)|^2 = 2(9 - \omega^2)(-2\omega) + 2 \cdot 16\omega$ is zero when $\omega = 0$ and when $(9 - \omega^2) = 8$, or $\omega = \pm 1$. Thus $\omega_r = 1$.

(b) The phase lag is the argument of $p(i\omega)$. This is $\frac{\pi}{4}$ when the real and imaginary parts are equal and positive. So $9 - \omega^2 = 4\omega$, or $\omega^2 + 4\omega - 9 = 0$, i.e. $(\omega + 2)^2 - 13$. This is zero when $\omega = -2 \pm \sqrt{13}$. Choose the + for a positive value: $\omega = \sqrt{13} - 2$.

5. By time-invariance, we can suppose that the input signal is $y = A\cos(\omega t)$. Replace y with $y_{cx} = Ae^{i\omega t}$. The equation is then $2\ddot{z} + \dot{z} + z = Ai\omega e^{i\omega t}$. $p(i\omega) = (1 - 2\omega^2) + i\omega$, so by the ERF $z_p = \dfrac{Ai\omega}{(1-2\omega^2) + i\omega} e^{i\omega t}$. So $H(\omega) = \dfrac{i\omega}{(1-2\omega^2) + i\omega}$. With $\omega = 1$, $H(1) = \frac{i}{-1+i} = \frac{1}{1+i}$, which has magnitude $g(1) = \frac{1}{\sqrt{2}}$. The phase lag is $-\text{Arg}\,(H(1)) = \frac{\pi}{4}$.

6. This is the real part of $\dfrac{d^3z}{dt^3} + z = e^{(-1+i)t}$. The characteristic polynomial is $p(s) = s^3 + 1$, and $p(-1 + i) = 2(1 + i) + 1 = 3 + 2i$. So $z_p = \dfrac{e^{(-1+i)t}}{3+2i} = e^{-t}\dfrac{3-2i}{13}e^{it}$, and $x_p = \text{Re}\,(z_p) = \frac{1}{13}e^{-t}(3\cos t + 2\sin t)$ (This can also be done using variation of paramters.)

7. (a) By linearity, $p(D)(\cos t - t) = p(D)\cos t - p(D)t = q(t) - q(t) = 0$. In fact $a(\cos t - t)$ will work for any a (except $a = 0$, since we wanted a nonzero solution).

(b) By linearity, we can add any homogeneous solution and get a new solution. If we start with $x_p = t$, we can add $x_h = 2(\cos t - t)$: $x = 2\cos t - t$.

(c) By time-invariance, $x(t - 1)$ will work, for any solution $x(t)$ of $p(D)x = q(t)$. So $t - 1$ and $\cos(t - 1)$ work, as does $a\cos(t - 1) + (1 - a)(t - 1)$ for any a.

Actually, LTI implies that if one sinusoidal function of circular frequency 1 is a solution of $p(D)x = 0$, then any sinusoidal function of circular frequency 1 is too, so there are even more choices of answers to all these questions.

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