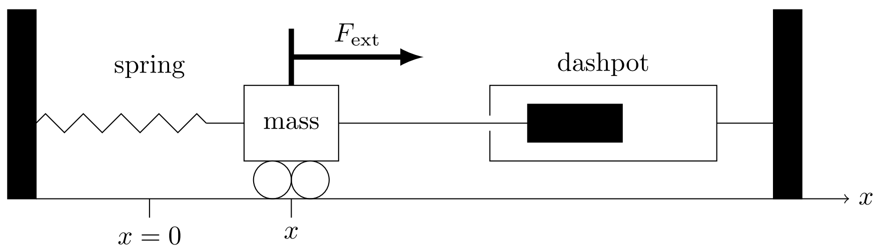
Damped Harmonic Oscillators

In the last session we modeled a spring-mass-dashpot system with the constant coefficient linear DE

$$m\ddot{x} + b\dot{x} + kx = F_{\text{ext}},$$

where m is the mass, b is the damping constant, k is the spring constant and $x(t)$ is the displacement of the mass from its equilibrium position.



We then assumed the external force $F_{\text{ext}} = 0$ and used the *characteristic equation* technique to solve the homogeneous equation

$$m\ddot{x} + b\dot{x} + kx = 0. \tag{1}$$

Restrictions on the coefficients: The algebra does not require any restrictions on m, b and k (except $m \neq 0$ so that the equation is genuinely second order). But, since this is a physical model, we will now require $m > 0$, $b \geq 0$ and $k > 0$.

The Damped Harmonic Oscillator: The undamped ($b = 0$) system has equation

$$m\ddot{x} + kx = 0.$$

At this point you should have memorized the solution *and* also be able to solve this equation using the characteristic roots. The solution is

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\cos(\omega t - \phi).$$

Here $\omega = \sqrt{k/m}$ and the solution is given in both rectangular and amplitude-phase form. The solution is always a sinusoid, which we consider a simple oscillation, and we call this system a **simple harmonic oscillator**.

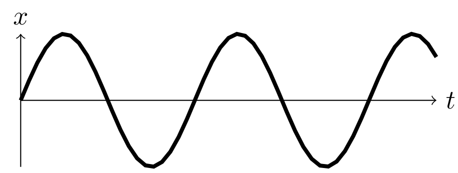


Figure 2: The output of a simple harmonic oscillator is a pure sinusoid.

When we add damping we call the system in (1) a **damped harmonic oscillator**. This is a much fancier sounding name than the spring-mass-dashpot. It emphasizes an important fact about using differential equations for modeling physical systems. It doesn't matter whether x measures position or current or some other quantity. Any system modeled by equation (1) will respond just like the spring-mass-dashpot; that is, all damped harmonic oscillators exhibit similar behavior. We will see an important example of this principle whe we study the case of an RLC electrical circuit.

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