Part II Problems and Solutions

Problem 1: *[Series RLC circuits; amplitude and phase] Open the Mathlet* `Series RLC Circuit`*. Here we will focus entirely on the current response, so it will be clearer if the check boxes labelled V_R, V_L, V_C, are left unchecked. But click twice on the I box, to make a green curve appear in the graphing window, representing the current through any point in the circuit as a function of time.*

The Mathlet uses the International System of Units, SI, formerly known as the mks (meter-kilogram-second) system. The equation

$$L\ddot{I} + R\dot{I} + (1/C)I = \dot{V}$$

is correct when:
the resistance R is measured in ohms, Ω,
the inductance L is measured in H, henries,
the capacitance C is measured in farads, F,
the voltage V is measured in volts, also denoted V,
the current I is measured in amperes, A.

The slider displays millihenries, mH (1 mH$= 10^{-3}$ H) and microfarads, μF (1μF$= 10^{-6}$ F), and milliseconds, ms (1 ms $= 10^{-3}$ sec).

The Mathlet studies a sinusoidal input signal $V(t) = V_0 \sin(\omega t)$. Play around with the various sliders and watch the effect on the (blue) voltage curve and the (green) current curve.

(a) *By experimenting, identify a few values of the system parameters R, L, C, V_0, ω, for which the current and the voltage are perfectly* in phase. *For example, if $L = 500$ mH and $\omega = 200$ radians/second, what values of R, C, and V_0 put I in phase with V?*

(b) *Now calculate the relationship between the system parameters which leads to I and V being in phase. Do your experiments align with your calculations?*

(c) *Set $R = 100\,\Omega$, $L = 1000$ mH, $C = 100\,\mu$F, $V_0 = 500$ V. Vary ω and watch the action. For what value of ω is the amplitude of $I(t)$ maximal? What is that amplitude (in amps)? What is the phase lag between the input signal, $V_0 \sin(\omega t)$, and the system response, $I(t)$, for that value of ω?*

(d) *Verify the three observations made in* **(c)** *computationally. You should be able to do this for general values of R, L, C, V_0.*

Solution: **(a)** It seems that C must be close to 50 μF. The values of V_0 and R don't seem to matter.

(b) Here is one of several ways to do this problem. We are looking at

$$L\ddot{I} + R\dot{I} + (1/C)I = V_0\omega\cos(\omega t).$$

To undersand its sinusoidal solution, make the complex replacement

$$L\ddot{z} + R\dot{z} + (1/C)z = V_0\omega e^{i\omega t},$$

so that $I_p = \mathrm{Re}(z_p)$. By the ERF, the exponential solution is $z_p = \dfrac{V_0\omega e^{i\omega t}}{p(i\omega)}$. To be in phase with $\sin(\omega t)$, the real part of this must be a positive multiple of $\sin(\omega t)$. This occurs precisely when the real part of $p(i\omega)$ is zero. $\mathrm{Re}\, p(i\omega) = (1/C) - L\omega^2$, so the relation is $1/C = L\omega^2$.

To check, when $L = 500$ mH$=$ .5 H and $\omega = 200$ rad/sec, the system response is in phase when $C = 1/(.5 \times (200)^2) = 50 \times 10^{-6}$ F = 50 μF.

(c) It seems that the maximal system response amplitude I_r occurs when $\omega = 100$ rad/sec, and that it is about 5 amps. Then the solution is in phase with the input voltage.

(d) In **(b)** we saw that the solution is the real part of $z_p = \dfrac{V_0\omega e^{i\omega t}}{p(i\omega)}$. The amplitude of this sinusoid is $\left|\dfrac{V_0\omega}{p(i\omega)}\right|$, which is maximal when its reciprocal $\left|\dfrac{(1/C - L\omega^2) + Ri\omega}{V_0\omega}\right| = \left|\left(\dfrac{1}{C\omega} - L\omega\right) + Ri\right|$ is minimal. The imaginary part here is constant, so as ω varies the complex number moves along the horizontal straight line with imaginary part R. The point on that line with minimal magnitude is Ri, which occurs when the real part is zero: $1/C\omega = L\omega$, or $\omega_r = 1/\sqrt{LC}$. The amplitude is then $I_r = g(\omega_r)V_0 = V_0/R$. It depends only on V_0 and R, not on L or C! Finally, this is the same as the condition for phase lag zero, so the phase lag at $\omega = \omega_r$ is zero.

With the given values $R = 100\ \Omega$, $L = 1$ H, $C = 10^{-4}$ F, $\omega_r = 100$ rad/sec, as observed. When $V_0 = 500$ V and $R = 100\ \Omega$, $I_r = 5$ Amps, as observed.

Problem 2: *AM Radio Tuning and LRC Circuits*

An LRC circuit can be modeled using the same DE as in the previous problem. Specifically,

$$LI'' + RI' + \frac{1}{C}I = E'.$$

Where I = current in amps, L = inductance in henries, R = resistance in ohms, C = capacitance in farads and E = input EMF in volts. Often the important output is the voltage drop V_R across the resistor. Ohm's law tells us $V_R = RI$. This gives us the DE

$$LV_R'' + RV_R' + \frac{1}{C}V_R = RE'.$$

(a) *Assume $E = E_0\cos(\omega t)$ and solve the DE for V_R in phase-amplitude form.*

(b) *Open the 'LRC Filter Applet'. This applet models an LRC circuit with input voltage a superposition of sine waves. Play with the applet –be sure to learn how to vary ω_1 and ω_2 by dragging the dots on the amplitude plot.*

Describe what happens to the amplitude response plot as L, R and C are varied.

(c) *An LRC circuit can be used as part of a simple AM radio tuner. In an AM radio broadcast the signal is given by $a\cos(\omega t)$ where ω is the 'carrier' frequency (between 530 and 1600 khz). To really carry information the amplitude a must vary –this is the amplitude modulation– but, we will ignore this right here.*

The range of values for this simple variable capacitor AM radio tuner is $L \approx .5$ microhenries, R is the resistance in the wire (very small) and C is between .02 and .2 microfarads. To keep things simple we will use different ranges however the idea is the same.

In the LRC Filter applet set $\omega_1 = 1$ and $w_2 = 4$ (set them as close as you can on your system). Set the input amplitudes a and b to 1. Find settings for L, R and C so that the output filters out the ω_2 part of the signal i.e. the output looks (a lot) like a sine wave of frequency ω_1. Give your values for L, R and C.

How does the quality of the filter change as you vary R?

(d) *An antenna on a radio picks up electomagnetic signals from all frequencies. It responds by outputing a signal consisting of voltages at each of these frequencies. This signal is used as input to a tuner circuit.*

Using the applet, set $L = 1$, $R = .5$. Now, vary C and then explain why a variable capacitor circuit could be used as an AM radio tuner.

(e) *Show that the natural frequency (undamped, unforced resonant frequency) of the system is $\omega_0 = 1/\sqrt{LC}$. Show that even with damping, i.e., $R > 0$, ω_0 is always the practical resonant frequency. (Hint: this can be done without calculus by writing $A(\omega)$ in the proper way.)*

Solution: (a) Here is the answer with very little comment. (Note we complexify the input before taking the derivative.)

For simplicity write V for V_R.

Complex DE: $L\tilde{V}'' + R\tilde{V}' + \frac{1}{C}\tilde{V} = (RE_0e^{i\omega t})' = i\omega RE_0e^{i\omega t}, \; V = \text{Re}(\tilde{V})$.

Char. polynomial: $P(iw) = \dfrac{1}{C} - L\omega^2 + Ri\omega$.

Note: $|P(i\omega)| = \dfrac{\sqrt{(1-LC\omega^2)^2 + (RC\omega)^2}}{C}$,

$$\frac{i}{P(i\omega)} = \frac{C}{(1-LC\omega^2)^2 + (RC\omega)^2}(RC\omega + i(1-LC\omega^2)).$$

Exp. Input Thm: $\tilde{V}_p = \dfrac{iR\omega E_0}{P(i\omega)} e^{i\omega t}$, $V_p = \text{Re}(\tilde{V}_p)$.

Amplitude-Phase Form: $\boxed{V_p = A\cos(\omega t - \phi)}$, where

$$A = \left|\frac{iR\omega E_0}{P(i\omega)}\right| = \frac{RC\omega E_0}{\sqrt{(1-LC\omega^2)^2 + (RC\omega)^2}} \Rightarrow \boxed{A = \frac{E_0}{\sqrt{\left(\frac{1-LC\omega^2}{RC\omega}\right)^2 + 1}}},$$

$$\boxed{\phi = -\text{Arg}\left(\frac{iR\omega E_0}{P(i\omega)}\right) = -\text{Arg}\left(\frac{i}{P(i\omega)}\right) = -\tan^{-1}\left(\frac{1-LC\omega^2}{RC\omega}\right)},$$

where, since the complex number $RC\omega + i(1-LC\omega^2)$ is in the 1st or 4th quadrants, ϕ is between $-\pi/2$ and $\pi/2$.

(b) Except for the fact that C corresponds to $1/k$ we get the same answer as the previous problems.

As L increases the amplitude peak moves to the left and the graph gets a little spikier.

As R decreases the peak doesn't move and the amplitude graph gets spikier.

As C increases the peak moves to the left.

(c) One possibility is $L = 3$, $C = .33$, $R = .4$. In any case, $LC = 1$. The smaller R is the less of the ω_2 frequency signal gets through. In general, the smaller the value of R the smaller the pass-band of the filter.

(d) As C varies the spike in the amplitude graph moves. Thus changing the frequency that can pass through the filter.

(e) Without damping or forcing the DE is $LI'' + \frac{1}{C}I = 0 \Rightarrow I'' + \frac{1}{LC}I = 0 \Rightarrow$ resonant frequency is $\omega_0 = 1/\sqrt{LC}$.

The boxed formula for A in part (a) shows that A is maximized when the term under the square root is minimized. This happens when the term in parentheses is 0, i.e. when $LC\omega^2 - 1 = 0 \Leftrightarrow \omega = 1/\sqrt{LC} = \omega_0$.