Superposition and the Integrating Factors Solution

1. Another Proof of the Superposition Principle

The superposition principle is so important a concept that it is worth reviewing yet again. Here we will use the integrating factors formula for the solution to first order linear ODE's to give another simple proof of this principle.

Recall, the standard first order linear ODE is

$$\dot{x} + p(t)x(t) = q(t). \qquad (1)$$

We derived the integrating factors solution

$$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)\,dt + c\right), \quad \text{where } u(t) = e^{\int p(t)\,dt}, \qquad (2)$$

and where the integral is any specific choice of the antiderivative and c is the constant of integration.

The superposition principle says that if:

x_1 is a solution to $\dot{x} + p(t)x(t) = q_1(t)$

and

x_2 is a solution to $\dot{x} + p(t)x(t) = q_2(t)$

then for any constants a and b, $ax_1 + bx_2$ is a solution to

$\dot{x} + p(t)x(t) = aq_1(t) + bq_2(t)$.

More briefly, we can write

$$q_1 \rightsquigarrow x_1 \text{ and } q_2 \rightsquigarrow x_2 \Rightarrow aq_1 + bq_2 \rightsquigarrow ax_1 + bx_2. \qquad (3)$$

To provide another way of thinking about this key principle, we'll rephrase it again in physical terms. If equation (1) models a physical situation and we consider $q(t)$ to be the input then the principle shown in (3) says superposition of inputs leads to superposition of outputs.

In fact, the proof takes only a few lines. Given the separate inputs $q_1(t)$ and $q_2(t)$, formula (2) gives the separate outputs

$$x_1(t) = \frac{1}{u(t)}\left(\int u(t)q_1(t)\,dt + c_1\right) \quad \text{and} \quad x_2(t) = \frac{1}{u(t)}\left(\int u(t)q_2(t)\,dt + c_2\right)$$

Now we use (2) to find the output for input $q = aq_1 + bq_2$. We will be able to choose any constant of integration, so, ahead of time, we choose the constant of integration to be of the form $c_1 + c_2$. Using the standard properties of integrals, the output is then

$$\begin{aligned} x(t) &= \frac{1}{u(t)} \left(\int u(t)(aq_1(t) + bq_2(t))\, dt + c_1 + c_2 \right) \\ &= \frac{a}{u(t)} \left(\int u(t)q_1(t)\, dt + c_1 \right) + \frac{b}{u(t)} \left(\int u(t)q_2(t)\, dt + c_2 \right) \\ &= ax_1(t) + bx_2(t) \quad \text{(which is what needed to be proved).} \end{aligned}$$

2. General = Particular + Homogeneous

In the general solution (2). we made a specific choice of the integral. By setting $c = 0$ this leads to a specific choice of the solution

$$x_p = \frac{1}{u(t)} \int u(t)q(t)\, dt.$$

We call x_p a **particular solution**, but this is a very poor name because there is nothing particularly particular about it. It is simply one specific solution. We could have chosen any other.

In the first note of this session we saw that the solution to the homogeneous equation (i.e., when $q(t) \equiv 0$). is related to the integrating factor u by $x_h(t) = 1/u(t)$. Using x_p and x_h we can rewrite the general solution (2) as

$$x(t) = x_p(t) + cx_h(t).$$

This tells us something interesting: one way to fully solve the inhomogeneous equation (1) is to first solve the homogeneous equation and then find any *one* solution, i.e., a *particular solution*, to the inhomogeneous equation.

We can use any method we want to find x_p. One method is the method of integrating factors, but for many equations we will have easier methods.

Example 1. Find the general solution to $\dot{x} + \frac{1}{t}x = t^2$ by finding an x_h and an x_p.

Solution. First we find x_h. The associated homogeneous equation is $\dot{x} + \frac{1}{t}x = 0$. This is separable and we easily solve it as follows.

Separate variables:	$\frac{dx}{x}$	$= -\frac{1}{t}\, dt$
Integrate:	$\ln \lvert x \rvert$	$= -\ln \lvert t \rvert + c$
Set $c = 0$, drop absolute values and exponentiate:	$x_h(t)$	$= \frac{1}{t}$.

Next, we use an integrating factor to find x_p. Formula (2) says $u(t) = e^{\int 1/t\,dt} = e^{\ln(t)} = t$. (Of course, we knew this since $u = 1/x_h$.) Thus, (again arbitrarily choosing the constant of integration to be 0)

$$x_p(t) = \frac{1}{u(t)} \int u(t)t^2\,dt = \frac{1}{t} \int t^3\,dt = \frac{t^3}{4}.$$

The general solution to the problem is therefore

$$x(t) = x_p(t) + cx_h(t) = \frac{t^3}{4} + \frac{c}{t}. \tag{4}$$

Notice, if we were a computer that didn't know any better, we might have chosen a different $x_p(t)$, say $x_p(t) = \frac{t^3}{4} + \frac{1}{t}$. We know this is a solution to our DE and so it has every right to be called a particular solution. In this case we would write our general solution as

$$x(t) = x_p(t) + cx_h(t) = \left(\frac{t^3}{4} + \frac{1}{t}\right) + \frac{c}{t}. \tag{5}$$

Equations (4) and (5) are both valid as general solutions. This is because both equations really represent a whole family of solutions (you get a different family member for each value of c) and each family contains the same set of solutions. For example, we get the same solution if we take $c = 5$ in equation (4) or if we take $c = 4$ in equation (5).

Example 2. Find the general solution to $\dot{x} + 2x = 4$.

Solution. The associated homogeneous equation is $\dot{x} + 2x = 0$. This models exponential decay and has a solution $x_h(t) = e^{-2t}$.

We'll use the method of optimism to find a particular solution. Since the right-hand side is a constant we guess a constant solution. By inspection we see that $x_p(t) = 2$ is one solution.

Combining the homogeneous and particular solutions, we get that the general solution is

$$x(t) = 2 + ce^{-2t}.$$

Example 3. Use the superposition principle to explain why $x(t) = x_p(t) + cx_h(t)$ is a solution to (1).

Solution. Let's use the language of inputs/outputs and call the right-hand side of (1) the input. The superposition principle says a superposition of inputs leads to a superposition of outputs.

That is, since x_p is a solution to the ODE with input $q(t)$ and x_h is a solution with input 0, we get the superposition $x_p + cx_h$ is a solution with input $q(t) + c \cdot 0 = q(t)$. This is exactly what we were asked to show.