Example: Heat Diffusion

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Here we will model heat diffusion with a first order linear ODE. We will solve the DE using the method of integrating factors.

Every summer I put my root beer in a cooler, but after a while it still gets warm. Let's model its temperature by an ODE. First we need to name the function that measures the temperature:

$$x(t) = \text{ root beer temperature at time } t.$$

The greater the temperature difference between inside and outside, the faster $x(t)$ changes. The simplest (linear) model of this is:

$$\dot{x}(t) = k(T_{\text{ext}}(t) - x(t)),$$

where $k > 0$ and $T_{\text{ext}}(t)$ is the external temperature.

One check that this makes sense (with $k > 0$) is to note that when the outside temperature T_{ext} is greater than the inside temperature $x(t)$, then $\dot{x}(t) > 0$ (since $k > 0$), so the temperature is increasing. Likewise,

$$T_{\text{ext}} < x(t) \Rightarrow \dot{x}(t) < 0 \Rightarrow \text{ the temperature is decreasing.}$$

This is indeed how heat behaves!

Rearranging the DE, we get the linear equation in standard form:

$$\dot{x} - kx = kT_{\text{ext}}(t). \tag{1}$$

This is **Newton's law of cooling**; k could depend upon t and we would still have a linear equation, but let's suppose that we are not watching the process for so long that the insulation of the cooler starts to break down!

Systems and signals analysis:

- The system is the cooler.
- The input signal is the external temperature $T_{\text{ext}}(t)$.
- The output signal or system response is $x(t)$, the temperature inside the cooler.

Note that the right-hand side of equation (1) is k times the input signal, not the input signal itself. As usual, what constitutes the input and output signals is a matter of the interpretation of the equation, not of the equation itself.

To take a specific example, let

$$x(0) = 32°\text{F}, \quad k = \frac{1}{3} \quad \text{and} \quad T_{\text{ext}}(t) = 60 + 6t \text{ in } °F,$$

where t denotes hours after 10AM. (That is, outside temperature is rising linearly.) We get the following differential equation and initial value:

$$\dot{x} + \frac{1}{3}x = 20 + 2t, \qquad x(0) = 32. \tag{2}$$

Solution. Again, until you can do it every time you should practice rederiving the integrating factors formula. Here we will use it directly.

Integrating factor: $u(t) = e^{\int \frac{1}{3}\,dt} = e^{\frac{1}{3}t}$ (choose any one possibility).

Solution:

$$\begin{aligned}
x(t) &= \frac{1}{u(t)}\left(\int u(t)\cdot(20+2t)\,dt + C\right) \\
&= e^{-t/3}\left(\int e^{t/3}(20+2t)\,dt + C\right) \\
&= e^{-t/3}(60e^{\frac{1}{3}t} + 6te^{\frac{1}{3}t} - 18e^{\frac{1}{3}t} + C) \qquad \text{(using integration by parts)} \\
x(t) &= 60 + 6t - 18 + Ce^{-\frac{1}{3}}t \\
&= 42 + 6t + Ce^{-\frac{1}{3}t}.
\end{aligned}$$

All that's left is to use the initial condition to find C. We plug in $t = 0$, $x(0) = 32$ and solve for C.

$$32 = x(0) = 42 + C \quad \Rightarrow \quad C = -10.$$

The equation describing the temperature inside my cooler is:

$$x(t) = 42 + 6t - 10e^{-t/3}.$$

We can use this to find how long it will take for my root beer to reach $60°F$. (I don't like it any warmer than that.) We need to solve

$$42 + 6t - 10e^{-t/3} = 60.$$

It is probably easiest to graph this function and read the correct value off the graph.

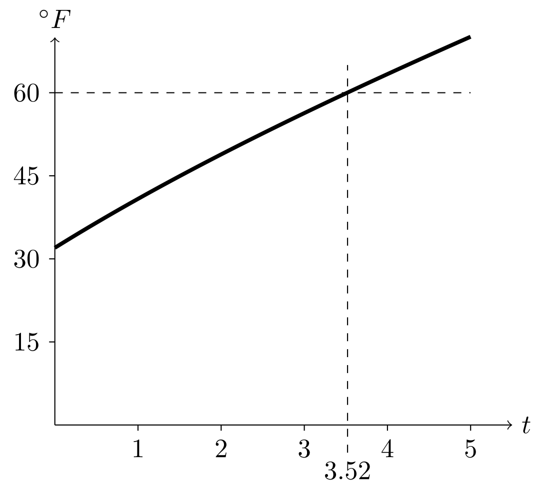

Fig. 1. I have roughly 3.5 hours to enjoy my root beer.

Remark: At this point the method of integrating factors is the only technique we have to solve this problem. For many problems it is the only technique, but as mentioned in the session introduction, we will eventually learn easier methods that work in this case.