

## Solutions to Linear First Order ODE's

### 1. First Order Linear Equations

In the previous session we learned that a first order linear *inhomogeneous* ODE for the unknown function  $x = x(t)$ , has the standard form

$$\dot{x} + p(t)x = q(t). \quad (1)$$

(To be precise we should require  $q(t)$  is not identically 0.)

We saw a bank example where  $q(t)$ , the rate money was deposited in the account, was called the input signal. We also saw an RC circuit example where the input signal was the voltage  $V(t)$  and  $q(t) = V'(t)$ .

A first order linear *homogeneous* ODE for  $x = x(t)$  has the standard form

$$\dot{x} + p(t)x = 0. \quad (2)$$

We will call this the **associated homogeneous equation** to the inhomogeneous equation (1)

In (2) the input signal is identically 0. We will call this the *null signal*. It corresponds to letting the system evolve in isolation without any external 'disturbance'.

- In the bank example: if there are no deposits and no withdrawals the input is 0.
- In the RC circuit example: if the power source is turned off and not providing any voltage increase then the input is 0.

### 2. Solutions to the Homogeneous Equations

The homogeneous linear equation (2) is separable. We can find the solution as follows:

- Separate variables:  $\frac{dx}{x} = -p(t)dt.$
- Integrate:  $\ln|x| = -\int p(t)dt + c_1.$  (We use  $c_1$  to save  $C$  for later.)
- Exponentiate:  $|x| = e^{c_1}e^{-\int p(t)dt}.$

- Rename  $e^{c_1}$  as  $C$ :  $|x| = C e^{-\int p(t)dt}$ ;  $C > 0$ .
- Drop the absolute value and recover the lost solution  $x(t) = 0$ :  
This gives the general solution to (2)

$$x(t) = C e^{-\int p(t)dt} \quad \text{where } C = \text{any value.} \quad (3)$$

A useful notation is to choose one specific solution to equation (2) and call it  $x_h(t)$ . Then the solution (3) shows the general solution to the equation is

$$x(t) = C x_h(t). \quad (4)$$

There is a subtle point here: formula (4) requires us to choose one solution to name  $x_h$ , but it doesn't matter which one we choose. We can say this somewhat awkwardly as choose an arbitrary specific solution.' A typical choice is to set the parameter  $C = 1$ , but this is not necessary.

**Example.** Solve  $\dot{x} + 2tx = 0$ .

**Solution.**

- Separate variables:  $\frac{dx}{x} = -2t dt$ .
- Integrate:  $\ln|x| = -\int 2t dt = -t^2 + c_1$ .
- Exponentiate and substitute  $C$  for  $e^{c_1}$ :  $|x| = e^{c_1} e^{-t^2} = C e^{-t^2}$ .
- Drop the absolute value and also recover the lost solution:  $x(t) = C e^{-t^2}$ .

In this example an obvious choice for  $x_h$  is  $x_h(t) = e^{-t^2}$ . It is clear the general solution to the example is

$$x(t) = C x_h(t) \quad \text{where } C = \text{any number.}$$

### 3. Solution to Inhomogeneous DE's Using Integrating Factors

We start with the integrating factors formula: the general solution to the inhomogeneous first order linear ODE (1) ( $\dot{x} + p(t)x = q(t)$ ) is

$$x(t) = \frac{1}{u(t)} \left( \int u(t)q(t)dt + C \right), \quad \text{where } u(t) = e^{\int p(t)dt}. \quad (5)$$

The function  $u$  is called an **integrating factor**.

This method, due to Euler, is easy to apply. We deduce it by the method of optimism, i.e., we introduce an integrating factor  $u$  and hope that it will help us.

**Proof:** We start with the product rule for differentiation

$$\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x.$$

and the equation (1):

$$\dot{x} + p(t)x = q(t).$$

Multiply both sides of the equation by some function  $u(t)$ , whose value we will determine later:

$$u\dot{x} + upx = uq. \quad (6)$$

In order to be able to apply the product rule we want the sum on the left hand side of the equation to have the form  $\frac{d}{dt}(ux) = u\dot{x} + \dot{u}x$ . There may be many functions  $u$  for which the left hand side has this form; we only need to find one of them. To do this, note that

$$\frac{d}{dt}(ux) = u\dot{x} + upx \Leftrightarrow u\dot{x} + \dot{u}x = u\dot{x} + upx \Leftrightarrow \dot{u} = up.$$

The last equation is a separable DE for the unknown function  $u$ :

$$\frac{du}{u} = p(t) dt$$

and so:

$$\begin{aligned} \ln |u| &= \int p(t) dt \\ u &= e^{\int p dt}. \end{aligned} \quad (7)$$

Remember, we are looking for just one  $u$ , so any choice of anti-derivative of  $p(t)$  in equation (7) will do.

Now replace the left-hand side of (6) by  $\frac{d}{dt}(ux)$  and solve for  $x$ :

$$\begin{aligned} u\dot{x} + upx &= uq \\ \frac{d}{dt}(ux) &= uq \\ u(t)x(t) &= \int u(t)q(t)dt + c \\ x(t) &= \frac{1}{u(t)} \left( \int u(t)q(t)dt + c \right) \end{aligned}$$

This last equation is exactly the formula (5) we want to prove.

**Example.** Solve the ODE  $\dot{x} + 2x = e^{3t}$  using the method of integrating factors.

**Solution.** Until you are sure you can rederive (5) in every case it is worthwhile practicing the method of integrating factors on the given differential equation. (At the end, we will model a solution that just plugs into (5).)

Multiply both sides by  $u$ :

$$u\dot{x} + 2u(t)x(t) = u(t) \cdot e^{3t}. \quad (8)$$

Next, find an integrating factor  $u$  so that the left-hand side is equal to  $\frac{d}{dt}(ux)$  (which equals  $u\dot{x} + \dot{u}x$ ).

$$\begin{aligned} u\dot{x} + \dot{u}x &= u\dot{x} + 2ux \\ \Rightarrow \quad \dot{u} &= 2u \\ u(t) &= e^{2t} \quad (\text{we choose any one } u \text{ that works}). \end{aligned}$$

Now substitute  $u(t) = e^{2t}$  into (8), then replace the left-hand side by  $\frac{d}{dt}(ux)$  and solve for  $x$ .

$$\begin{aligned} \frac{d}{dt}(e^{2t}x) &= e^{2t}e^{3t} \\ \Rightarrow \quad e^{2t}x &= \frac{1}{5}e^{5t} + C \quad (\text{integrate the previous equation}) \\ \Rightarrow \quad x(t) &= \frac{1}{5}e^{3t} + Ce^{-2t} \quad (\text{solve for } x(t)). \end{aligned}$$

Here is a model of the same solution using (5) directly.

Integrating factor:  $u(t) = e^{\int 2dt} = e^{2t}$  (choose any one possibility).

Solution:

$$\begin{aligned} x(t) &= \frac{1}{u(t)} \int u(t)e^{3t} dt \\ &= e^{-2t} \int e^{5t} dt \\ &= e^{-2t} \left( \frac{1}{5}e^{5t} + C \right) \\ &= \frac{1}{5}e^{3t} + Ce^{-2t}. \end{aligned}$$

#### 4. Comparing the Integrating Factor $u$ and $x_h$

Recall that in section 2 we fixed one solution to the homogeneous equation (2) and called it  $x_h$ . The formula for  $x_h$  is

$$x_h(t) = e^{-\int p(t) dt},$$

where we can pick any one choice for the antiderivative. Comparing this with the formula for the integrating factor

$$u = e^{\int p(t) dt}$$

we get the following relationship between the two functions:

$$x_h(t) = \frac{1}{u(t)}.$$

The solution to the homogeneous equation (or for short *the homogeneous solution*)  $x_h$  will play an extremely prominent role in the rest of the course.

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