

18.03SC Practice Problems 5

First order Linear ODEs: Integrating factors

Solution Suggestions

1. Around here, the ocean experiences tides. About twice a day the ocean level rises and falls by several feet. This is why small boats are often tied up to floating docks.

A salt pond on Cape Cod is connected to the ocean by means of a narrow channel. This problem will explore how the water level in the pond varies.

In roughest terms, the water level in the bay increases, over a small time interval, by an amount which is proportional to (1) the difference between the ocean level and the bay level and (2) the length of the small time interval.

(a) Write $y(t)$ for the height of the water in the ocean, measured against some zero mark, and $x(t)$ for the height of the water in the bay, measured against the same mark. Set up the first order linear equation that describes this model. What is the “system” here? What part of the ODE represents it? What function is the “input signal”? What is the “output signal”?

We are given that over a small time interval, Δx is proportional to $(y - x)\Delta t$. Let us call the constant of proportionality k . Then our differential equation is $\frac{dx}{dt} = k(y(t) - x(t))$. In standard form, this is

$$\dot{x} + kx = ky$$

The system is the bay/channel/ocean complex – the geography, and its behavior is described by the left hand side.

The input signal is the height of the water in the ocean, and the output signal is the height of the water in the bay.

(b) Assume that the tide is high exactly every 4π hours – not a bad approximation. Suppose that the ocean height is given by $y(t) = \cos(\omega t)$ (in meters and hours). What value does ω take?

When $t = 0$, $\cos(\omega 0) = \cos 0 = 1$ is a maximum, so the tide is high. The tide becomes high again when $\cos(\omega t) = 1$ again, i.e., when $\omega t = 2\pi$. At this time, $t = 4\pi$, so $\omega = 1/2$.

If you recognized that the period here is 4π and ω is the angular frequency, you could have computed directly $\omega = \frac{2\pi}{4\pi} = \frac{1}{2}$.

(c) Find a solution of your differential equation using integrating factors. You may find the following integral useful – a consequence, you may recall, of two integration by parts:

$$\int e^{kt} \cos(\omega t) dt = \frac{1}{k^2 + \omega^2} e^{kt} (k \cos(\omega t) + \omega \sin(\omega t)) + c$$

We want a function u such that $u\dot{x} + ukx = \frac{d}{dt}(ux)$. Solving $\dot{u} = ku$ gives $u = e^{kt}$.

Multiplying our differential equation by u gives $\frac{d}{dt}(ux) = uky$, or $x = (e^{kt})^{-1} \int e^{kt} ky dt =$

$e^{-kt} \int e^{kt} k \cos(t/2) dt$. Evaluate the integral involved by using the formula given above for $\omega = 1/2$, and obtain the general solution

$$x(t) = \frac{2k \sin(t/2) + 4k^2 \cos(t/2)}{1 + 4k^2} + ce^{-kt}.$$

A basic sanity check is to make sure the solution we found satisfies the original equation. Here the derivative is $\dot{x} = k \frac{\cos(t/2) - 2k \sin(t/2)}{1 + 4k^2} - cke^{-kt}$, so $\dot{x} + kx = k \frac{\cos(t/2) + 4k^2 \cos(t/2)}{1 + 4k^2} = ky$.

Pause for a moment to examine the form of the general solution. It is the sum of a periodic part and a decaying term ($k \geq 0$). The decaying term is *transient*, and, after a long time, no matter what the initial conditions may be, the height of the water in the bay is modeled by the periodic part, which is the *steady state* of the solution. Note that if $k = 0$, the channel is closed.

(d) Your solution probably had the form $a \cos(\omega t) + b \sin(\omega t)$ for some constants a, b . Find it a second time by substituting this expression into the differential equation and solving for a and b .

Let's take the $c = 0$ solution from above. Then $x = a \cos(t/2) + b \sin(t/2)$ and $\dot{x} = -a/2 \sin(t/2) + b/2 \cos(t/2)$. We find that $\dot{x} + kx = (ak + b/2) \cos(t/2) + (bk - a/2) \sin(t/2) = k \cos(t/2) = ky$. When $t = 0$, we have $ak + b/2 = k$, so $b = -2ka + 2k$. When $t = \pi$, we have $bk - a/2 = 0$, so $a = 2bk = -4k^2a + 4k^2$. Then $a = \frac{4k^2}{1+4k^2}$ and $b = \frac{2k}{1+4k^2}$, which matches what we found above.

2. In a previous session we studied the direction field of the differential equation

$$\frac{dy}{dx} = x - 2y.$$

Now find the general solution of this equation analytically.

When we studied the direction field we found a straight line solution. Does it occur within the general solution here?

Plot it and several other solutions, using your expression for the general solution. Is there a funnel?

Find the particular solution with initial value $y(0) = 1$.

We use integrating factors. In standard form, $y' + 2y = x$, so $u = e^{2x}$. Then $y = u^{-1} \int ux dx = e^{-2x} \int xe^{2x} dx$. Integrating by parts yields $\int xe^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{2} \int e^{2x} dx = \frac{x}{2} e^{2x} - \frac{1}{4} e^{2x} + c$. Therefore, $y = x/2 - 1/4 + ce^{-2x}$.

The straight line solution occurs when $c = 0$. There is a funnel coming from the exponentially decaying term.

Here is a reproduction of the screenshot from the Isoclines Mathlet from the session on graphical methods..

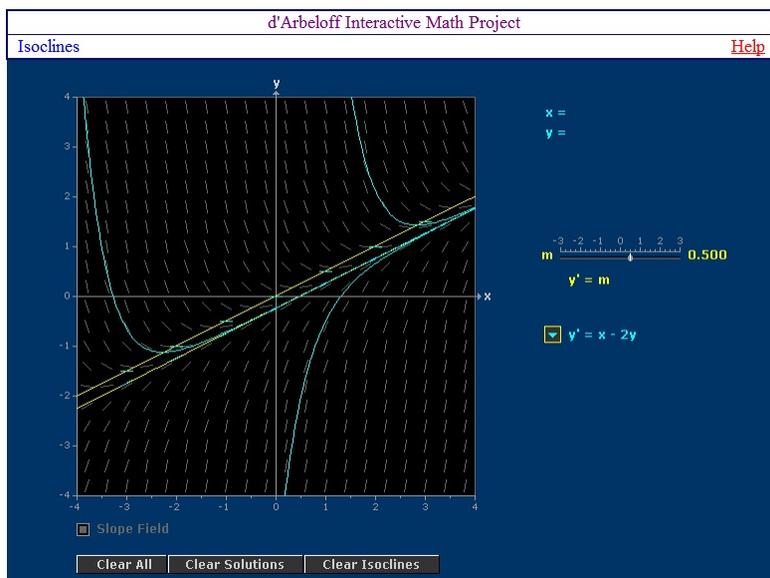


Figure 1: A plot of the straight line solution and several others for $\frac{dy}{dx} = x - 2y$. The particular solution with initial value $y(0) = 1$ has $1 = 0/2 - 1/4 + c \cdot 1 = -1/4 + c$, i.e. $c = 5/4$. That is, it is the solution $y = x/2 - 1/4 + 5/4e^{-2x}$.

3. Find the general solution of

$$x^2 \frac{dy}{dx} + 2xy = \sin(2x)$$

by recognizing the left hand side as the derivative of a product.

We see that $x^2 y' + 2xy = (x^2 y)'$. Thus, $x^2 y = -\frac{1}{2} \cos(2x) + c$, and $y = cx^{-2} - \frac{\cos(2x)}{2x^2}$.

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