Part I Problems and Solutions

For each of the next three problems, solve the given linear DE. Give the general solution, and also the specific solution satisfying the initial condition.

Problem 1:

$$\frac{dy}{dx} + y = 2 \qquad\qquad y(0) = 0$$

Solution: Integrating factor $\rho = e^x = e^{\int 1 dx}$

General solution $y = 2 + ce^{-x}$

Specific solution $y = 2(1 - e^{-x})$

Problem 2: $xy' - y = x$ *and* $x(1) = 7$

Solution: First need to bring to the form $\frac{dy}{dx} + P(x)y = Q(x)$ in order to compute the IF $\rho = e^{\int P(x)dx}$.

So $y' - \frac{1}{x}y = 1 \to \rho = e^{-\int \frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$y = x\left(c + \int \frac{1}{x} \cdot 1dx\right) \to y = x(c + \ln x)$ (general solution)

$y(1) = 1(c + 0) = c = 7 \to y = x(7 + \ln x)$ (specific solution)

Problem 3: $y' = 1 + x + y + xy, \qquad y(0) = 0$

Solution: First rewrite as $y' - (1 + x)y = 1 + x$.

IF: $\rho = e^{-\int (1+x)dx} = e^{-(x+x^2/2)}$

General solution: $y = e^{x+x^2/2}\left(c + \int (1 + x)e^{-(x+x^2/2)}\right)$ so

$$y = -1 + ce^{x+x^2/2}$$

using $\int (1 + x)e^{-(x+x^2/2)}dx = -e^{-x+x^2/2}$.

Specific solution: $y(0) = -1 + c \cdot 1 = 0 \to c = 1$ so $y = e^{x+x^2/2} - 1$

Problem 4: *Water flows into and out of a 100,000 liter (ℓ) reservoir at a constant rate of 10 ℓ/min. The reservoir initially contains pure water, but then the water coming in has a concentration of 10 grams/liter of a certain pollutant. The reservoir is well-stirred so that the concentration of pollutant in it is uniform at all times.*

a) Set up the DE for the concentration $c = c(t)$ of salt in the reservoir at time t. Specify units.

b) Solve for $c(t)$ with the given initial condition, and graph the solution c vs. t.

c) How long will it take for the concentration of salt to be 5 $\frac{g}{\ell}$?

d) What happens in the long run?

Solution: Let $x = x(t)$ be the *amount* of salt in the reservoir at time t, with x in grams and t in minutes. Then $c(t) = x(t)/V$ or $c(t) = 10^{-5}x(t)$ in $\frac{g}{\ell}$. We will work with $x(t)$, and then get $c(t)$ at the end.

a) $\frac{dx}{dt}$ = salt rate in - salt rate out = net rate of change.

Rate in is $10\frac{g}{\ell} \times 10\frac{\ell}{min} = 10^2\frac{g}{min}$

Rate out is $10\frac{\ell}{min} \times \frac{x(t)}{V} = 10\frac{\ell}{min} \times \frac{x(t)}{10^5}\frac{g}{\ell} = 10^{-4}x(t)\frac{g}{min}$

Thus,

$$\frac{dx}{dt} = 10^2 - 10^{-4}x$$

in $\frac{g}{min}$. The initial condition is $x(0) = 0$.

b) Can use linear or separable method.
Using separable: (Exercise: solve using linear method and compare results)

$$\begin{aligned}\frac{dx}{10^6 - x} &= 10^{-4}dt \\ -\ln(10^6 - x) &= 10^{-4}t + c \\ 10^6 - x &= Ce^{-10^{-4}t} \\ x &= 10^6 - Ce^{-10^{-4}t}\end{aligned}$$

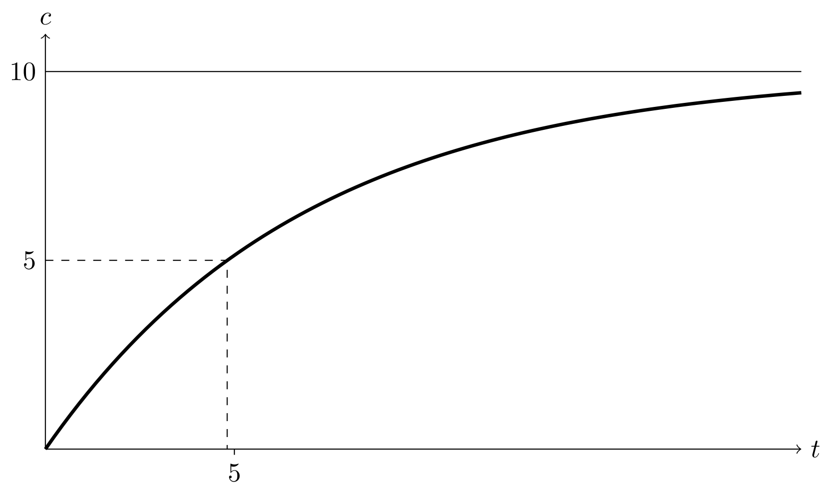
$x(0) = 0$: $x(0) = 10^6 - c = 0 \rightarrow c = 10^6$.

$$x(t) = 10^6(1 - e^{-10^{-4}t}) \qquad \text{ingrams.}$$

Thus,

$$c(t) = 10^{-5} \cdot x(t) = 10(1 - e^{-10^{-4}t})$$

in $\frac{g}{\ell}$, with t in minutes



c) $c(t) = 5 = 10(1 - e^{-10^{-4}t} \rightarrow \frac{1}{2} = 1 - e^{-10^{-4}t} \rightarrow -10^{-4}t = -\ln 2 \rightarrow t = 10^4 \ln 2 \approx 6931.5$ min, or $t \approx 4.81$ days.

d) $c(t) \rightarrow 10\frac{g}{\ell}$ = the input concentration as $t \rightarrow \infty$.