Part II Problems and Solutions

Problem 1: *[Linear models] Scrooge McDuck wants to set up a trust fund for his nephew Don. He has fool-proof investments which make a constant interest rate of I, measured in units of* $(\text{years})^{-1}$ *(so $I = 0.05$ means 5% per year), and he proposes to dole out the money to his profligate nephew at a constant rate q dollars per year.*

(a) *Model this process by a differential equation. (Use the symbols I and q, rather than specific values for them.) Explain your steps.*

(b) *Then find the general solution to this differential equation.*

(c) *Now take $I = 0.05$. If Uncle Scrooge wanted to fund the trust so as to provide his nephew with \$1000 per* month *in perpetuity, while maintaining a constant balance in the fund, how much should he invest?*

(d) *But in fact Uncle Scrooge wants to teach his nephew the virtues of self-reliance, and so plans on having the trust fund run entirely out of money in exactly twenty years. If he wants to give his nephew \$1000 per month, how much should he fund the trust with at the outset? Give the answer to the nearest penny (as Scrooge would insist on).*

Solution: **(a)** Pick a letter to denote the number of years after the fund is set up—say t. Pick a letter to denote the function of t giving the value of the fund at time t—say x. In a small time interval from t to $t + \Delta t$, the fund increases in value by $Ix(t)\Delta t$, but decreases in value by $q\Delta t$: $x(t + \Delta t) - x(t) \simeq Ix(t)\Delta t - q\Delta t$. Divide by Δt and take the limit: $\dot{x} = Ix - q$.

(b) Separate: $dx/(Ix - q) = dt$. Integrate: $I^{-1} \ln |Ix - q| + c_1 = t + c_2$. Amalgamate constants and multiply by I: $\ln |Ix - q| = It + c$. Exponentiate: $|Ix - q| = e^c e^{It}$. Eliminate the absolute value and reintroduce the lost solution: $Ix - q = Ce^{It}$. Solve for x: $x = (q/I) + Ce^{It}$ (where this C is the earlier one divided by I).

(c) Constant trust value means $\dot{x} = 0$, which says $Ix = q$ or $x = q/I$. So with $q = 12{,}000$ dollars/year and $I = 0.05$, $x = \$240{,}000$. (If Scrooge socks away more than this, then the trust fund could pay out the \$1000/month and still grow. But this wouldn't be Scrooge.)

(d) We want to find the constant of integration which makes $x(T) = 0$, where $T = 20$: $0 = x(T) = (q/I) + Ce^{IT}$, or $C = -(q/I)e^{-IT}$. Thus $x = (q/I)(1 - e^{-IT}e^{It})$. Now we can set $t = 0$ to find the required initial value of the trust: $x(0) = (q/I)(1 - e^{-IT})$. With $T = 20$ and $I = 0.05$, $1 - e^{-IT} = 1 - e^{-1} \simeq 0.63212056$. Thus the initial funding is about 63% of what it was in **(c)**: $x(0) \simeq (\$240{,}000)(.63212056) \simeq \$151{,}708.93$.

Problem 2: *[Solutions to linear equations] Almost all the radon in the world today was created within the past week or so by a chain of radioactive decays beginning mainly from uranium, which has been part of the earth since it was formed. This cascade of decay-*

ing elements is quite common, and in this problem we study a "toy model" in which the numbers work out decently. This is about Tatooine, a small world endowed with unusual elements.

A certain isotope of Startium, symbol St, decays with a half-life t_S. Strangely enough, it decays with equal probability into a certain isotope of either Midium, Mi, or into the little known stable element Endium. Midium is also radioactive, and decays with half-life t_M into Endium. All the St was in the star-stuff that condensed into Tatooine, and all the Mi and En arise from the decay route described. Also, $t_M \neq t_S$.

Use the notation $x(t)$, $y(t)$, and $z(t)$, for the amount of St, Mi, and En on Tatooine, in units so that $x(0) = 1$. Also, assume $y(0) = 0$ and $z(0) = 0$.

(a) *Make rough sketches of graphs of x, y, z, as functions of t. What are the limiting values as $t \to \infty$?*

(b) *Write down the differential equations controlling x, y, and z. Be sure to express the constants that occur in these equations correctly in terms of the relevant decay constants. Use the notation σ (Greek letter sigma) for the decay constant for St and μ (Greek letter mu) for the decay constant for Mi. Your first step is to relate σ to t_S and μ to t_M. A check on your answers: the sum $x + y + z$ is constant, and so we should have $\dot{x} + \dot{y} + \dot{z} = 0$.*

(c) *Solve these equations, successively, for x, y, and z.*

(d) *At what time does the quantity of Midium peak? (This will depend upon σ and μ.)*

(e) *Suppose that instead of $x(0) = 1$, we had $x(0) = 2$. What change will this make to $x(t)$, $y(t)$, and $z(t)$?*

(f) *Unrelated question: Suppose that $x(t) = e^t$ is a solution to the differential equation $t\dot{x} + 2x = q(t)$. What is $q(t)$? What is the general solution?*

Solution: **(b)** Startium obeys the natural decay equation, $\dot{x} = -\sigma x$, with solution $x = x(0)e^{-\sigma t}$. To relate σ to its half-life, solve for it in $x(0)/2 = x(0)e^{-\sigma t_S}$ to find $\sigma = (\ln 2)/t_S$. Similarly, $\mu = (\ln 2)/t_M$.

Midium decays as well, but in each small time interval gets half the decayed Startium added: so $y(t + \Delta t) \simeq -\mu y(t)\Delta t + \frac{1}{2}\sigma x(t)\Delta t$. Thus $\dot{y} = -\mu y + \frac{1}{2}\sigma x$. Endium receives half the decayed Startium and all the decayed Midium: $\dot{z} = \frac{1}{2}\sigma x + \mu y$. Adding these three equations gives $\dot{x} + \dot{y} + \dot{z} = 0$.

(c) Using $x(0) = 1$, we know that $x = e^{-\sigma t}$. Thus $\dot{y} + \mu y = \frac{1}{2}\sigma e^{-\sigma t}$. An integrating factor is given by $e^{\mu t}$: $\frac{d}{dt}(e^{\mu t}y) = \frac{1}{2}\sigma e^{(\mu-\sigma)t}$. Integrating, $e^{\mu t}y = \frac{1}{2}\frac{\sigma}{\mu-\sigma}e^{(\mu-\sigma)t} + c$ or $y = \frac{1}{2}\frac{\sigma}{\mu-\sigma}e^{-\sigma t} + ce^{-\mu t}$. The initial condition is $y(0) = 0$, so $c = -\frac{1}{2}\frac{\sigma}{\mu-\sigma}$: $y = \frac{1}{2}\frac{\sigma}{\mu-\sigma}(e^{-\sigma t} - e^{-\mu t})$.

We could solve for z in the same way, but it's easier to calculate $z = 1 - x - y = 1 +$

$\frac{\sigma/2-\mu}{\mu-\sigma}e^{-\sigma t} + \frac{\sigma/2}{\mu-\sigma}e^{-\mu t}$

(d) From the differential equation for y, we know that a critical point occurs when $\mu y = \frac{1}{2}\sigma e^{-\sigma t}$. Substitute the value for y: $\mu \frac{1}{2}\frac{\sigma}{\mu-\sigma}(e^{-\sigma t} - e^{-\mu t}) = \frac{1}{2}\sigma e^{-\sigma t}$. Some algebra leads to $\sigma e^{-\sigma t} = \mu e^{-\mu t}$, so $e^{(\mu-\sigma)t} = \mu/\sigma$, so $t_{\text{max}} = \frac{\ln\mu - \ln\sigma}{\mu-\sigma}$.

(e) Everything gets doubled.

(f) If $x = e^t$ then $q(t) = t\dot{x} + 2x = te^t + 2e^t = (t+2)e^t$. The associated homogeneous equation is $t\dot{x} + 2x = 0$, which is separable: $dx/x = -2dt/t$, so $\ln|x| = -2\ln|t| + c = \ln(t^{-2}) + c$ and $x = C/t^2$. So the general solution of the original equation is $e^t + C/t^2$.