18.03SC Differential Equations, Fall 2011
Transcript – Sinusoidal Functions

PROFESSOR: So today we're going to take a look at sinusoidal functions. And specifically, we're going to use complex numbers to get a handle on sinusoidal functions. The reason is complex numbers provide a robust way of analyzing sinusoidal functions.

So specifically, we're interested in this function, e to the i omega t divided by 2 plus 3i. And we're asked to write the real part of this function using polar form and then rectangular form. And then secondly, we're asked several properties of this function, the real part of this function. What's the circular frequency? What's the amplitude? And what's the phase lag? And then lastly, we're asked to sketch the real part of this function versus time.

So I'll let you take a look at this and try it for yourself. And I'll come back in a moment.

Hi everyone. Welcome back. OK. So let's take a look at this problem.

So we're asked to write e to i omega t divided by 2 plus 3i. We're asked to write the real part of this function, using polar form and then also rectangular form. So I'll first start off with polar form. And if we take a look at this function, we see that the numerator is already written in polar form. So just recall that polar form is of the form r e to the i theta.

Meanwhile, the denominator is written in Cartesian form, or rectangular form. So we need to convert this into polar form. So what we can do is we can write 2 plus 3i. We can combine these two into a modulus. And the modulus is going to be the square root of 2 squared plus 3 squared. So this is the modulus of the complex number.

And then we have e to the i phi. And the angle phi we can deduce by writing a triangle, which is over to the right by two units, and then it's up two units. And this is the angle phi. So we see from the triangle that the tangent of phi is equal to 3/2.

OK. So putting the pieces together, we have e to the i omega t divided by root 13 e to i phi. And the beautiful thing about polar form is that when we divide and multiply these complex exponentials, it just turns into a simple addition or subtraction of the phases.

So we have 1 over root 13, e to the i omega t minus phi. So we've successfully combined all the terms into one polar term. And then now we're asked to compute the real part. And here we use Euler's formula. So just to recall Euler's formula, we have e to the i theta. Then this is equal to cosine theta plus i sine theta. OK?

So at the end of the day, we're interested in the real part-- and I'm going to use this notation Re with a curly bracket to denote the real part-- of 1 over root 13. And I'll use Euler's formula on e to the i omega t minus phi. So we have cosine omega t minus phi plus i sine omega t minus phi. Sorry. 1 over root 13 is multiplying both the sign and the cosine.

And we see that the second term is purely imaginary. So when we take the real part of a real number plus an imaginary number, we just are left with the real number at the end of the day. So the answer we're looking for is 1 over root 13 cosine omega t minus phi, where from before phi is the arctangent of 3/2. And phi can also be thought of the angle being between zero and pi over 2.

OK. So this concludes the polar form computation. Secondly, we also have to compute this using a rectangular form calculation. So let's do the calculation in rectangular form. And for this, I'm going to use Euler's formula to expand out the numerator.

So first, we want to take the real part of the numerator is cosine omega t plus i sine omega t. The denominator was 2 plus 3i. And typically what we want to do is we want to turn the denominator into an entirely real number. So what we do is we multiply the top and bottom by the complex conjugate of the denominator.

So for example, I have 2 plus 3i. So I'm going to multiply the top and bottom by 2 minus 3i. OK. And when I do this, you'll note that when we multiply out two plus 3i and 2 minus 3i we have a difference of squares. So we're left with 2 times 2 is 4. The cross terms cancel. And then we have 3i times negative 3i. That's going to give us negative 9i squared, which is plus 9.

So the denominator is going to be 4 plus 9. And then the numerator-- I'll write out all the terms just for completeness-- it'll be 2 times cosine omega t plus 3 sine omega t. And then now we have some imaginary terms, which are negative 3 cosine omega t and plus 2 sine omega t. OK?

But now we're going to take the real part of this complex function. And if we take a look at it, this first term is entirely real. This second term is entirely complex because we're multiplying by an i. So when we take the real part, at the end of the day, the second term is going to drop out.

So when the dust settles, we have 1/13 2 times cosine omega t plus 3 sine omega t. OK? So this concludes Part A.

For Part B, we're asked several questions about this sinusoidal function when written in its real form, specifically, what's the circular frequency. And the circular frequency is just the frequency that this function oscillates at. And we can get this from looking at the rectangular form or the polar form. And it's simply just going to be omega. So note how this function oscillates with frequency omega.

Secondly, we're asked about the amplitude. And if we take a look at it, the amplitude is a little tricky to get from the rectangular form, but it's completely apparent from the polar form. So if we take a look at the polar form, we have 1 over root 13 cosine omega t minus phi.

And cosine oscillates between plus 1 and minus 1. So this function takes on a maximum value of plus 1 over root 13 and a maximum negative value of negative 1 over root 13. So the amplitude of this function is just going to be 1 over root 13.

And then lastly, the phase lag. This is the angle that the cosine is picked up and shifted over. So again, we can get this directly from the polar form.

Meanwhile, it's a little less obvious what this angle is when we take a look at the rectangular form. So in our case, the phase lag is just going to be phi, which is 10 inverse 3/2. OK? So this concludes Part B.

And then lastly, for Part C we're just asked to sketch this diagram. And again, the most direct way at sketching this function is through the polar form. So I'll write time on this axis. And if we just recall that a cosine might look something like this. So this is a cosine function.

And in our case, we're looking at plotting 1 over root 13 cosine omega t minus phi. So we're interested in a cosine which has an amplitude of 1 over root 13 and a maximum negative value of minus 1 over root 13 whose frequency is omega. So there's a quick way on finding zeros. So if the zero of a regular cosine is at pi over 2, then if we have an angular frequency of omega, the zero here is going to be pi over 2 omega. And this zero is going to be at 3 pi over 2 omega.

And lastly, this is not necessarily the cosine of our function. Our function is this cosine. However it must be picked up and shifted over by some phase lag phi. So in our case, the function we're actually looking for is some shifted cosine, which looks like this. And this distance here is phi over omega. OK?

So this just gives us a quick sketch of this function. And again, we note that we were able to plot this using the polar form very quickly. And the reason is because the polar form gives us the amplitude, the circular frequency, and the phase lag. Whereas if we were to take a look at the equivalent rectangular form, it's a little less obvious. The rectangular form is actually the sum of a sine and a cosine. And unless you can just add sines and cosines in your head very quickly, it's a little less obvious on how to come up with this picture.

OK. So just to summarize, we've used complex functions to analyze sinusoidal functions. And we've used polar form and rectangular form to get a handle on the complex functions. And specifically, complex functions are a very good way to represent sinusoidal functions. And just to reiterate, the reason is that they contain a lot of information that we can graphically handle and turn into a sketch. So I'll just conclude there, and I'll see you next time.