The Sinusoidal Identity

The sum of two sinusoidal functions of the same frequency is another sinusoidal function with that frequency! For any real constants a and b,

$$a\cos(\omega t) + b\sin(\omega t) = A\cos(\omega t - \phi) \tag{1}$$

where A and ϕ can be described in at least two ways:

$$A = \sqrt{a^2 + b^2}, \quad \phi = \tan^{-1}\frac{b}{a}; \tag{2}$$

$$a + bi = Ae^{i\phi}. \tag{3}$$

Conversely, we have

$$a = A\cos(\phi) \text{ and } b = A\sin(\phi). \tag{4}$$

Geometrically this is summarized by the triangle in the figure below.

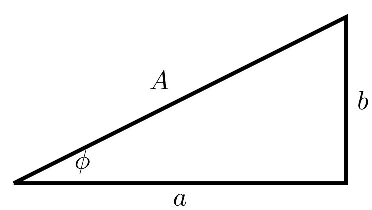

Fig. 1. $a + bi = Ae^{i\phi}$.

One proof of (1) is a simple application of the cosine addition formula

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta).$$

We will now give an equivalent proof using Euler's formula and complex arithmetic: The triangle in Figure 1 is the standard polar coordinates triangle. It shows $a + ib = Ae^{i\phi}$ or $a - ib = Ae^{-i\phi}$. Thus

$$\begin{aligned}
A\cos(\omega t - \phi) &= \mathrm{Re}(Ae^{i(\omega t - \phi)}) \\
&= \mathrm{Re}(e^{i\omega t} \cdot Ae^{-i\phi}) \\
&= \mathrm{Re}((\cos(\omega t) + i\sin(\omega t)) \cdot (a - ib)) \\
&= \mathrm{Re}(a\cos(\omega t) + b\sin(\omega t) + i(a\sin(\omega t) - b\cos(\omega t))) \\
&= a\cos(\omega t) + b\sin(\omega t).
\end{aligned}$$

We should stress the importance of the trigonometric identity (1). It shows that *any* linear combination of $\cos(\omega t)$ and $\sin(\omega t)$ is not only periodic of period $\frac{2\pi}{\omega}$, but is also sinusoidal. If you try to add $\cos(\omega t)$ to $\sin(\omega t)$ "by hand", you will probably agree that this is not at all obvious.

We will call $A\cos(\omega t - \phi)$ **amplitude-phase form** and $a\cos(\omega t) + b\sin(\omega t)$ **rectangular** or **Cartesian form**. You should be familiar with amplitude-phase form; we usually prefer it because both amplitude and phase have geometric and physical meaning for us.