## Part II Problems and Solutions

**Problem 1: [Linear vs Nonlinear]** *Still working with the equation* $\dot{y} = (1-y)y - a$ *with* $a = \frac{3}{16}$*, let* $y_0$ *be the stable critical point. Write* $u = y - y_0$ *for the population excess over equilibrium (so* $u < 0$ *if the population is less than the equilibrium value).*

**(a)** *Rewrite the differential equation as a differential equation for* $u$*. Check that the new equation is again autonomous and that* $u = 0$ *is a critical point for it.*

**(b)** *For small* $u$ *we can neglect higher powers of* $u$ *(such as* $u^2$*). This process is "linearization near equilibrium." What is the linearized equation near* $u = 0$*? What is the general solution of this linear autonomous equation?*

**(c)** *At least in this case, when solutions of the original autonomous equation get near equilibrium, they are well modeled by solutions of the linearization. Give an approximation of* $y$ *near equilibrium. Use it to answer this question: if* $y(10) - y_0 = b$*, estimate* $y(11)$ *and* $y(12)$*.*

**(d)** *Suppose that the linear equation* $\dot{x} + p(t)x = q(t)$ *is autonomous. What can you say about* $p(t)$ *and* $q(t)$*?*

**Solution: (a)** $y_0 = 3/4$, from **(b)** above: $y = u + \frac{3}{4}$, so $1 - y = \frac{1}{4} - u$ and $\dot{u} = (\frac{1}{4} - u)(u - \frac{3}{4}) - \frac{3}{16} = -\frac{1}{2}u - u^2$. No explicit time dependence, so autonomous; and if $u = 0$ then $\dot{u} = 0$.

**(b)** The linearized equation is $\dot{u} = -\frac{1}{2}u$. The general solution to this is $u = ce^{-t/2}$.

**(c)** Thus $y$ is well approximated by $\frac{3}{4} + ce^{-t/2}$: the population decays, or relaxes, exponentially (with decay rate $\frac{1}{2}$) to the equilibrium value.

**(d)** Both $p(t)$ and $q(t)$ must be constants.