First order Linear Differential Equations

To start we will define *first order linear equations* by their form. Soon, we will understand them by their properties. In particular, you should be on the lookout for the statement of the *superposition principle* and in later sessions a conceptual definition of linearity.

Definition. The general **first order linear ODE** in the unknown function $x = x(t)$ has the form:

$$A(t)\frac{dx}{dt} + B(t)x(t) = C(t). \qquad (1)$$

As long as $A(t) \neq 0$ we can simplify the equation by dividing by $A(t)$.

$$\frac{dx}{dt} + p(t)x(t) = q(t) \qquad (2)$$

We'll call (2) the **standard form** for a first order linear ODE.

1. Terminology and Notation

The functions $A(t)$, $B(t)$ in (1), and $p(t)$ in (2), are called the **coefficients** of the ODE. If A and B (or p) are constants (i.e. do not depend on the variable t) we say the equation is a **constant coefficient** DE.

We use the familiar notations y' or $\dot{y}$ for the derivative of y. With some exceptions, we'll use $\dot{y} = \frac{dy}{dt}$ to mean the derivative with respect to time and y' for derivatives with respect to some other variable, e.g. $y' = \frac{dy}{dx}$. If there is any danger of confusion we'll revert to the unambiguous Liebnitz notatiation: $\frac{dy}{dt}$, $\frac{dy}{dx}$, etc.

2. Homogeneous/Inhomogeneous

If $C(t) = 0$ in (1) the resulting equation:

$$A(t)\dot{x} + B(t)x = 0$$

is called **homogeneous**[1]. Likewise, in standard form, $\dot{x} + p(t)x = 0$ is homogeneous. Otherwise the equation is **inhomogeneous**.

[1] Homogeneous is not the same as homogenous (or homogenized). The syllable "ge" has a long e and is stressed in homogeneous, while the syllable "mo" is stressed in homogenous.

3. Examples

We will give two examples where we construct models that give first order linear ODE's.

Example 1. In session 1 we modeled an oryx population x with natural growth rate k and harvest rate h:

$$\dot{x} = kx - h, \text{ or } \dot{x} - kx = -h.$$

Fig. 1. Oryx. Image courtesy of Cape Town Craig on flickr.

We repeat the argument leading to this model. We start with the population $x(t)$ at time t. A natural growth rate k means that after a short time Δt we would expect there to be approximately $kx(t)\Delta t$ more oryx. However, in that same time $h\Delta t$ oryx are harvested. So we have the net change in the oryx population:

$$\Delta x \approx kx(t)\Delta t - h\Delta t \qquad \Longrightarrow \qquad \frac{\Delta x}{\Delta t} \approx kx(t) - h.$$

Now, letting the time interval Δt approach 0 we get the ODE $\frac{dx}{dt} = kx(t) - h$.

Note: if the rates k and h are not constant, but vary with time, the modeling process will lead to the same differential equation:

$$\frac{dx}{dt} = k(t)x(t) - h(t) \quad \text{or} \quad \frac{dx}{dt} - k(t)x(t) = -h(t).$$

Example 2. (Bank account)
I have a bank account. It has $x(t)$ dollars in it, i.e., x is a function of time. I can deposit money in the account and make withdrawals from it. The bank pays me interest for the money in my account. We will call the interest rate r, it has units of $(\text{year})^{-1}$.

In the old days a bank would pay interest at the end of the month on the balance at the beginning of the month. We can model this mathematically.

With $\Delta t = 1/12$, the statement at the end of the month will read:

$$x(t + \Delta t) = x(t) + rx(t)\Delta t + [\text{deposits} - \text{withdrawals between } t \text{ and } t + \Delta t].$$

These days r is typically very small, say 1%/year = 0.01/year. And, you don't get 1% each month! You get 1/12 of that.

You can think of a withdrawal as a negative deposit, so I will call everything a 'deposit' and allow the sign to positive or negative.

Nowadays interest is usually computed daily. This is a step on the path to the enlightenment afforded by calculus, in which $\Delta t \to 0$ and the interest is computed continuously.

In order to reach enlightenment, I want to record deposits minus withdrawals as a *rate*, in dollars per year. Suppose I contribute \$100 sometime every month, and make no withdrawals. My total deposits up to time t, that is, my *cumulative total* deposit $Q(t)$ has a graph like the following figure.

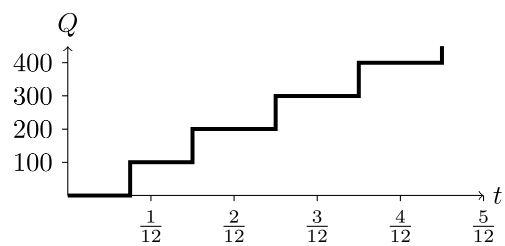

Fig. 2. With periodic deposits $Q(t)$ is a step function.

In keeping with letting $\Delta t \to 0$, we should imagine that I am making this contribution continually at the constant rate of \$1200/year. Then the graph of $Q(t)$ is a straight line with slope 1200, shown in figure below. In this case, the derivative $Q'(t) = q(t)$ is constant.

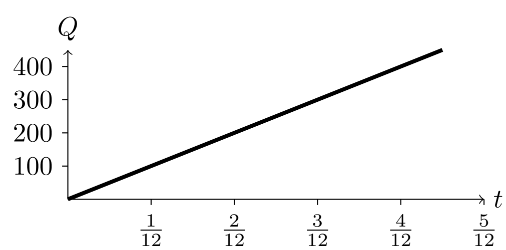

Fig. 3. With continuous deposits the graph of $Q(t)$ is a straight line.

In general, say I deposit at the rate of $q(t)$ dollars per year. The value of $q(t)$ might vary over time, and might be negative from time to time, because, with our convention, withdrawals are merely negative deposits.

So, (assuming $q(t)$ is continuous),

$$x(t + \Delta t) \approx x(t) + rx(t)\Delta t + q(t)\Delta t.$$

Now subtract $x(t)$ and divide by Δt:

$$\frac{x(t + \Delta t) - x(t)}{\Delta t} \approx rx + q$$

Next, let the interest period Δt tend to zero:

$$\dot{x} = rx + q.$$

Note: $q(t)$ can certainly vary in time. The interest rate can too. In fact the interest rate might depend upon x as well: a larger account will probably earn a better interest rate. Neither feature affects the derivation of this equation, but if r does depend upon x as well as t , then the equation we are looking at is no longer linear. So, for this example, let's say $r = r(t)$ and $q = q(t)$.

We can put the linear ODE into standard form:

$$\dot{x} - r(t)x = q(t).$$