Response to Discontinuous Input

We will continue looking at the constant coefficient first order linear DE

$$\dot{y} + ky = q(t).$$

It has the integrating factors solution

$$y = e^{-kt}\left(\int e^{kt}q(t)dt + c\right). \tag{1}$$

In this note we want to do an example where the input $q(t)$ is discontinuous.

The most basic discontinuous function is the **unit-step function** at a point a, defined by:

$$u_a(t) = \begin{cases} 0 & t < a \\ 1 & t > a. \end{cases} \tag{2}$$

(We leave its value at a undefined, though some books give it the value 0 there, others the value 1 there.)

Example 1. We'll look again at Newton's law of cooling and my root beer cooler:

$$\dot{y} + ky = kf(t),$$

where, $y(t)$ is the temperature inside the cooler and $f(t)$ is the temperature of the air. It's a nice, cool morning with constant temperature. Suddenly the sun comes out and the air warms up to a higher constant temperature. What's the response of my cooler to this signal?

We'll assume the sun comes out at time $t = a$, my cooler starts at $t = 0$ with temperature 0 and (somewhat idealized) the air temperature jumps instantly from 0 to 20 at time $t = a$. So $f(t) = 20\, u_a(t)$ and our IVP is

$$\dot{y} + ky = k20u_a(t), \quad y(0) = 0.$$

Solution. For $t < a$ we have the input is 0. Since $y(0) = 0$, the response is $y(t) = 0$.

For $t \geq a$ the DE becomes $\dot{y} + ky = 20k$ with $y(a) = 0$. The solution (which we have found before) is $y(t) = 20 + ce^{-kt}$. Now we use the initial condition $y(a) = 0$ to the find the value of c. We get $c = -20e^{ka}$, so $y(t) = 20 - 20e^{ka}e^{-kt}$ for $t \geq a$.

We can now assemble the results for $t < a$ and $t \geq a$ into one expression; for the latter, we also put the exponent into a more suggestive form.

$$\text{input} = 20u_a(t) \quad \longrightarrow \quad \text{response} = y(t) = \begin{cases} 0 & 0 < t < a; \\ 20 - 20e^{-k(t-a)} & t \geq a. \end{cases} \tag{3}$$

Note that the response is just the translation a units to the right of the response to the unit-step input at 0.

Our next example continues the temperature model with a different discontinuous input. In this case, the physical input is an external bath which is initially ice-water at 0 degrees, then replaced by water held at a fixed temperature for a time interval, then replaced once more by ice-water. To model the input we need the **unit box function** on $[a, b]$:

$$u_{ab} = \begin{cases} 1 & a \leq t \leq b \\ 0 & \text{otherwise} \end{cases} \qquad 0 \leq a < b; \tag{4}$$

Example 2. Find the response of the system

$$\dot{y} + ky = kq, \quad \text{with IC } y(0) = 0$$

to input $q(t) = 20u_{ab}(t)$.

Solution. There are at least three ways to do this:

a) Express u_{ab} as a sum of unit step functions and use (3) together with superposition of inputs;

b) Use the function u_{ab} directly in a definite integral expression for the response;

c) Find the response in two steps: first use (3) to get the response $y(t)$ for the input $u_a(t)$; this will be valid up till the point $t = b$.

   Then, to continue the response for values $t > b$, evaluate $y(b)$ and find the response for $t > b$ to the input 0, with initial condition $y(b)$.

We will follow (c), leaving the first two as exercises.

By (3), the response to the input $u_a(t)$ is:

$$y(t) = \begin{cases} 0 & 0 \leq t < a \\ 20 - 20e^{-k(t-a)} & t \geq a. \end{cases}$$

This is valid up to $t = b$, since $u_{ab}(t) = u_a(t)$ for $t \leq b$. Evaluating at b,

$$y(b) = 20 - 20e^{-k(b-a)}. \tag{5}$$

For $t > b$ we have $u_{ab} = 0$, so the DE is just $\dot{y} + ky = 0$. This models exponential decay (our most important DE) and we know the solution:

$$y(t) = ce^{-kt}. \tag{6}$$

We determine c from the initial value (5). Equating the initial values $y(b)$ from (5) and (6), we get:

$$ce^{-kb} = 20 - 20e^{-kb+ka}$$

from which:

$$c = 20e^{kb} - 20e^{ka}.$$

By (6):

$$y(t) = 20(e^{kb} - e^{ka})e^{-kt}, \quad t \geq b. \tag{7}$$

After combining exponents in (7) to give an alternative form for the response we assemble the parts, getting:

$$y(t) = \begin{cases} 0 & 0 \leq t \leq a \\ 20 - 20e^{-k(t-a)} & a < t < b \\ 20e^{-k(t-b)} - 20e^{-k(t-a)} & t \geq b. \end{cases} \tag{8}$$

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