18.03SC Differential Equations, Fall 2011

Transcript – Lecture 7

This is also written in the form, it's the k that's on the right hand side. Actually, I found that source is of considerable difficulty. And, in general, it is. For these, the temperature concentration model, it's natural to have the k on the right-hand side, and to separate out the (q)e as part of it. Another model for which that's true is mixing, as I think I will show you on Monday. On the other hand, there are some common first-order models for which it's not a natural way to separate things out. Examples would be the RC circuit, radioactive decay, stuff like that.

So, this is not a universal utility. But I thought that that form of writing it was a sufficient utility to make a special case, and I emphasize it very heavily in the nodes. Let's look at the equation. And, this form will be good enough, the y'. When you solve it, let me remind you how the solutions look, because that explains the terminology. The solution looks like, after you have done the integrating factor and multiplied through, and integrated both sides, in short, what you're supposed to do, the solution looks like y equals, there's the term e^(-k) out front times an integral which you can either make definite or indefinite, according to your preference.

Integral(q(t) e^(kt) dt), it will help you to remember the opposite signs if you think that when q is a constant, one, for example, you want these two guys to cancel out and produce a constant solution. That's a good way to remember that the signs have to be opposite. But, I don't encourage you to remember the formula at all. It's just a convenient thing for me to be able to use right now.

And then, there's the other term, which comes by putting up the arbitrary constant explicitly, ce^(-kt). So, you could either write it this way, where this is somewhat vague, or you could make it definite by putting a zero here and a t there, and change the dummy variable inside according to the way the notes tell you to do it. Now, when you do this, and if k is positive, that's absolutely essential, only when that is so, then this term, as I told you a week or so ago, this term goes to zero because k is positive as t goes to infinity. So, this goes to zero as t goes, and it doesn't matter what c is, as t goes to infinity. This term stays some sort of function. And so, this term is called the steady-state or long-term solution, or it's called both, a long-term solution.

And this, which disappears, gets smaller and smaller as time goes on, is therefore called the transient because it disappears at the time increases to infinity. So, this part uses the initial condition, uses the initial value. Let's call it y(0), assuming that you started the initial value, t, when t was equal to zero, which is a common thing to do, although of course not necessary.

The starting value appears in this term. This one is just some function. Now, the general picture or the way that looks is, the steady-state solution will be some solution like, I don't know, like that, let's say. So, that's a steady-state solution, the SSS. Well, what do the other guys look like? Well, the steady-state solution has this starting point. Other solutions can have any of these other starting points. So, in the beginning, they won't look like the steady-state solution. But, we know that as time goes on, they must approach it because this term represents the difference between the solution and the steady-state solution.

So, this term is going to zero. And therefore, whatever these guys do to start out with, after a while they must follow the steady-state solution more and more closely. They must, in short, be asymptotic to it. So, the solutions to any equation of that form will look like this. Up here, maybe it started at 127. That's okay. After a while, it's going to start approaching that green curve. Of course, they won't cross each other. That's the rock star, and these are the groupies trying to get close to it. Now, but something follows from that picture. Which is the steady-state solution? What, in short, is so special about this green curve? All these other white solution curves have that same property, the same property that all the other white curves and the green curve, too, are trying to get close to them.

In other words, there is nothing special about the green curve. It's just that they all want to get close to each other. And therefore, though you can write a formula like this, there isn't one steady-state solution. There are many. Now, this produces vagueness. You talk about the steady-state solution; which one are you talking about? I have no answer to that; the usual answer is whichever one looks simplest. Normally, the one that will look simplest is the one where c is zero. But, if this is a peculiar function, it might be that for some other value of c, you get an even simpler expression. So, the steady-state solution: about the best I can see, either you integrate that, don't use an arbitrary constant, and use what you get, or pick the simplest.

Pick the value of c, which gives you the simplest answer. Pick the simplest function, and that's what usually called the steady-state solution. Now, from that point of view, what I'm calling the input in this input response point of view, which we are going to be using, by the way, constantly, well, pretty much all term long, but certainly for the next month or so, I'm constantly going to be coming back to it. The input is the q(t). In other words, it seems rather peculiar. But the input is the right-hand side of the equation of the differential equation. And the reason is because I'm always thinking of the temperature model.

The external water bath at temperature T external, the internal thing here, the problem is, given this function, the external water bath temperature is driving, so to speak, the temperature of the inside. And therefore, the input is the temperature of the water bath. I don't like the word output, although it would be the natural thing because this temperature doesn't look like an output. Anyone might be willing to say, yeah, you are inputting the value of the temperature here. This, it's more likely, the normal term is response. This thing, this plus the water bath, is a little system. And the response of the system, i.e. the change in the internal temperature is governed by the driving external temperature. So, the input is q(t), and the response of the system is the solution to the differential equation.

Now, if the thing is special, as it's going to be for most of this period, it has that special form, then I'm going to, I really want to call (q)e the input. I want to call q sub e the input, and there is no standard way of doing that, although there's a most common way. So, I'm just calling it the physical input, in other words, the temperature input, or the concentration input. And, that will be my (q)e (t) and by the subscript e, you will understand that I'm writing it in that form and thinking of this model, or concentration model, or mixing model as I will show you on Monday.

By the way, this is often handled, I mean, how would you handle this to get rid of a k? Well, divide through by k. So, this equation is often, in the literature, written this way: (1/k)*y' + y = (q)e (t). They call it q of t, not (q)e of t because they've gotten rid of this funny factor. But I will continue to call it (q)e. So, in other words, and this part this is just, frankly, called the input. It doesn't say physical or anything. And, this is the solution, it's then the response, and this funny coefficient of y', that's not in standard linear form, is it, anymore? But, it's a standard form if you want to do this input response analysis. So, this is also a way of writing the equation.

I'm not going to use it because how many standard forms could this poor little course absorb? I'll stick to that one. Okay, you have, then, the superposition principle, which I don't think I'm going to-- the solution, which solution? Well, normally it means any solution, or in other words, the steady-state solution. Now, notice that terminology only makes sense if k > 0. And, in fact, there is nothing like the picture, the picture doesn't look at all like this if k is negative, and therefore, the terms would steady state, transient would be totally inappropriate if k were negative. So, this assumes definitely that k has to be greater than zero.

Otherwise, no. So, I'll call this the physical input. And then, you have the superposition principle, which I really can't improve upon what's written in the notes, this superposition of inputs. Whether they are physical inputs or nonphysical inputs, if the input q of t produces the response, y of t, and q2 (t) --> y2 (t), -- -- then a simple calculation with the differential equation shows you that by, so to speak, adding, that the sum of these two, I stated it very generally in the notes but it corresponds, we will have as the response y1, the steady-state response y1 + y2, and a constant times y1.

That's an expression, essentially, of the linear, it uses the fact that the special form of the equation, and we will have a very efficient and elegant way of seeing this when we study higher order equations. For now, I will just, the little calculation that's done in the notes will suffice for first-order equations. If you don't have a complicated equation, there's no point in making a fuss over proofs using it. But essentially, it uses the fact that the equation is linear.

Or, that's bad, so linearity of the ODE. In other words, it's a consequence of the fact that the equation looks the way it does. And, something like this would not, in any sense, be true if the equation, for example, had here a y^2 instead of t. Everything I'm saying this period would be total nonsense and totally inapplicable. Now, today, what I wanted to discuss was, what's in the notes that I gave you today, which is, what happens when the physical input is trigonometric? For certain reasons, that's the most important case there is. It's because of the existence of what are called Fourier series, and there are a couple of words about them. That's something we will be studying in about three weeks or so.

What's going on, roughly, is that, so I'm going to take the equation in the form y' + ky = k q(e) (t), and the input that I'm interested in is when this is a simple one that you use on the visual that you did about two points worth of work for handing in today, cos(omega t). So, if you like, k here. So, the (q)e is cosine omega t. That was the physical input. And, omega, as you know, is, you have to be careful when you use the word frequency. I assume you got this from physics class all last semester. But anyway, just to remind you, there's a whole yoga of five or six terms that go whenever you're talking about trigonometric functions.

Instead of giving a long explanation, the end of the second page of the notes just gives you a reference list of what you are expected to know for 18.03 and physics as well, with a brief one or two line description of what each of those means. So, think of it as something to refer back to if you have forgotten. But, omega is what's called the angular frequency or the circular frequency. It's somewhat misleading to call it the frequency, although I probably will.

It's the angular frequency. It's, in other words, it's the number of complete oscillations. This cos(omega t) is going up and down right? So, a complete oscillation as it goes down and then returns to where it started. That's a complete oscillation. This is only half an oscillation because you didn't give it a chance to get back. Okay, so the number of complete oscillations in how much time, well, in two pi, in the distance, two pi on the t-axis in the interval of length two pi because, for example, if omega is one, cos(t) takes two pi to repeat itself, right?

If omega were two, it would repeat itself. It would make two complete oscillations in the interval, two pi. So, it's what happens to the interval, two pi, not what happens in the time interval, one, which is the natural meaning of the word frequency. There's always this factor of two pi that floats around to make all of your formulas and solutions incorrect. Okay, now, so, what I'm out to do is, the problem is for the physical input, (q)e cos(omega t), find the response. In other words, solve the differential equation.

In short, for the visual that you looked at, I think I've forgot the colors now. The input was in green, maybe, but I do remember that the response was in yellow. I think I remember that. So, find the response, yellow, and the input was, what color was it, green? Blue, blue. Light blue. Okay, so we've got to solve the differential equation. Now, it's a question of how I'm going to solve the differential equation. I'm going to use complex numbers throughout, A because that's the way it's usually done. B, to give you practice using complex numbers, and I don't think I need any other reasons.

So, I'm going to use complex numbers. I'm going to complexify. To use complex numbers, what you do is complexification of the problem. So, I'm going to complexify the problem, turn it into the domain of complex numbers. So, take the differential equation, turn it into a differential equation involving complex numbers, solve that, and then go back to the real domain to get the answer, since it's easier to integrate exponentials. And therefore, try to introduce, try to change the trigonometric functions into complex exponentials, simply because the work will be easier to do. All right, so let's do it.

To change this differential equation, remember, I've got cos(omega t) here. I'm going to use the fact that e^(i omega t), Euler's formula, that the real part of it is cos(omega t). So, I'm going to view this as the real part of this complex function. But, I will throw at the imaginary part, too, since at one point we will need it. Now, what is the equation, then, that it's going to turn into?

The complexified equation is going to be y' + ky = k e^(i omega t). Now, I have a problem because y, here, in this equation, y means the real function which solves that problem. I therefore cannot continue to call this y because I want y to be a real function. I have to change its name. Since this is complex function on the right-hand side, I will have to expect a complex solution to the differential equation.

I'm going to call that complex solution y tilda. Now, that's what I would also use as the designation for the variable. So, y tilda is the complex solution. And, it's going to have the form y1 + i*y2. It's going to be the complex solution. And now, what I say is, so, solve it. Find this complex solution. So, find the program is to find y tilde, -- -- that's the complex solution. And then I say, all you have to do is take the real part of that, and that will answer the original problem. Then, y1, that's the real part of it, right? It's a function, you know, like this is cosine plus sine, as it was over here, it will naturally be something different. It will be something different, but that part of it, the real part will solve the original problem, the original, real, ODE.

Now, you will say, you expect us to believe that? Well, yes, in fact. I think we've got a lot to do, so since the argument for this is given in the nodes, so, read this in the notes. It only takes a line or two of standard work with differentiation. So, read in the notes the argument for that, why that's so. It just amounts to separating real and imaginary parts. Okay, so let's, now, solve this. Since that's our program, all we have to find is the solution. Well, just use integrating factors and just do it. So, the integrating factor will be, what, e to the, I don't want to use that formula.

So, the integrating factor will be e^(kt) is the integrating factor. If I multiply through both sides by the integrating factor, then the left-hand side will become y e to the kt, the way it always does, prime, [y~ e^(kt)]', and the right-hand side will be, now I'm going to start combining exponentials. It will be k e^(i(omega t + k)). I'm going to write that k + omega t.

i omega t + k. Thank you. i omega t plus k, or k + i omega t. kt? Sorry. So, it's k e^(i omega t) e^(kt). So, that's (k + i omega)t. Sorry. So, y~ e^(kt) is k divided by, now I integrate this, so it essentially reproduces itself, except you have to put down on the bottom k + i omega.

I'll take the final step. What's y tilda equals, see, when you do it this way, then you don't get a messy looking formula that you substitute into and that is scary looking. This is never scary. Now, I'm going to do two things simultaneously. First of all, here, if I multiply, after I get the answer, I'm going to want to multiply it by e^(-kt), right, to solve for y tilda.

If I multiply this by e to the negative kt, then that just gets rid of the k that I put in, and left back with e to the i omega t. So, that side is easy. All that is left is e^(i omega t). Now, what's interesting is this thing out here, k + i omega. I'm going to take a typical step of scaling it. And you scale it. I'm going to divide the top and bottom by k.

And, what does that produce? 1 / (1 + i(omega / k)). What I've done is take these two separate constants, and shown that the critical thing is their ratio. Okay, now, what I have to do now is take the real part. Now, there are two ways to do this. There are two ways to do this. Both are instructive. So, there are two methods. I have a multiplication. The problem is, of course, that these two things are multiplied together. And, one of them is, essentially, in Cartesian form, and the other is, essentially, in polar form. You have to make a decision.

Either go polar, it sounds like go postal, doesn't it, or worse, like a bear, savage, attack it savagely, which that's a very good, aggressive attitude to have when doing a problem, or we can go Cartesian. Going polar is a little faster, and I think it's what's done in the nodes. The notes to do both of these. They just do the first. On the other hand, they give you a formula, which is the critical thing that you will need to go Cartesian. I hope I can do both of them if we sort of hurry along. How do I go polar? To go polar, what you want to do is express this thing in polar form.

Now, one of the things I didn't emphasize enough, probably, when I talked to you about complex numbers last time is, so I will remind you, which saves my conscience and doesn't hurt yours, suppose you have alpha as a complex number. See, this complex number is a reciprocal. The good number is what's down below. Unfortunately, it's downstairs. You should know, like you know the back of your hand, which nobody knows, one over alpha.

So that's the form. The number I'm interested in, that coefficient, it is of the form one over alpha. (1 / alpha)*alpha = 1. And, from that, it follows, first of all, if I take absolute values, if the absolute value of one over alpha times the absolute value of this is equal to one, so, this is equal to one over the absolute value of alpha. I think you all knew that. I'm a little less certain you knew how to take care of the angles.

How about the argument? Well, the argument of the angle, in other words, the angle of one over alpha plus, because when you multiply, angles add. Remember that. Plus, the angle associated with alpha has to be the angle associated with one. But what's that? One is out here. What's the angle of one? Zero. Therefore, the argument, the absolute value of this thing is want over the absolute value. That's easy. And, you should know that the argument of want over alpha is equal to minus the argument of alpha. So, when you take reciprocal, the angle turns into its negative. Okay, I'm going to use that now, because my aim is to turn this into polar form. So, let's do that someplace, I guess here.

So, I want the polar form for 1 / (1 + i(omega / k)). Okay, I will draw a picture. Here's one. Here is (omega / k). Let's call this angle phi. It's a natural thing to call it. It's a right triangle, of course. Okay, now, this is going to be a complex number times e to an angle. Now, what's the angle going to be? Well, this is a complex number, the angle for the complex number. So, the argument of the complex number, 1 + i(omega / k) is how much? Well, there's the complex number one plus i over (1 + i) / (1 + i(omega / k)). Its angle is phi. So, the argument of this is phi, and therefore, the argument of its reciprocal is negative phi

So, it's e^(-i phi). And, what's A? A is one over the absolute value of that complex number. Well, the absolute value of this complex number is 1 + (omega/k)^2. So, the A = 1 / sqrt(1 + (omega / k)^2) * e^(-i phi).

See, I did that. That's a critical step. You must turn that coefficient. If you want to go polar, you must turn is that coefficient, write that coefficient in the polar form. And for that, you need these basic facts about, draw the complex number, draw its angle, and so on and so forth. And now, what's there for the solution? Once you've done that, the work is over. What's the complex solution? The complex solution is this. I've just found the polar form for this. So, I multiply it by e^(i omega t), which means these things add. So, it's equal to A e^(i omega t - i phi).

Or, in other words, the coefficient is 1 / sqrt(1 + (omega / k)^2). And, this is e to the, see if I get it right, now. And finally, now, what's the answer to our real problem? y1: the real answer. I mean: the really real answer. What is it? Well, this is a real number. So, I simply reproduce that as the coefficient out front. And for the other part, I want the real part of that. But you can write that down instantly. So, let's recopy the coefficient. And then, I want just the real part of this. Well, this is e to the i times some crazy angle.

So, the real part is the cosine of that crazy angle. So, it's the cos(omega t - phi). And, if somebody says, yeah, well, okay, I got the omega k, I know what that is. That came from the problem, the driving frequency, driving angular frequency. That was omega, and k, I guess, k was the conductivity, the thing which told you how quickly the heat that penetrated the walls of the little inner chamber.

So, that's okay, but what's this phi? Well, the best way to get phi is just to draw that picture, but if you want a formula for phi, phi will be, well, I guess from the picture, it's phi = atan(omega / k), which I don't have to put in. So, it's this number, phi, in reference to this function. See, if the phi weren't there, this would be cos(omega t), and we all know what that looks like. The phi is called the phase lag or phase delay, something like that, the phase lag of the function. What does it represent? It represents, let me draw you a picture.

Let's draw the picture like this. Here's cos(omega t). Now, regular cosine would look sort of like that. But, I will indicate that the angular frequency is not one by making my cosine squinchy up a little too much. Everybody can tell that that's the cosine on a limp axis, something for Salvador Dali, okay. So, there's cosine of something. So, what was it? Blue? I don't have blue. Yes, I have blue. Okay, so now you will know what I'm talking about because this looks just like the screen on your computer when you put in the visual for this. Frequency: your response order one. So, this is cos(omega t).

Now, how will cos(omega t - phi) look? Well, it'll be moved over. Let's, for example, suppose phi were pi over two. Now, where's pi over two on the picture? Well, what I do is cos(omega t) minus this. I move it over by one, so that this point becomes that one, and it looks like, the site will look like this. In other words, I shove it over by, so this is the point where omega t = pi / 2.

It's not the value of t. It's not the value of t. It's the value of omega t. And, when I do that, then the blue curve has been shoved over one quarter of its total cycle, and that turns it, of course, into the sine curve, which I hope I can draw. So, this goes up to there, and then, it's got to get back through. Let me stop there while I'm ahead. So, this is sin(omega t), the yellow thing, but that's also, in another life, cos(omega t - pi/2). The main thing is you don't subtract, the pi over two is not being subtracted from the t.

It's being subtracted from the whole expression, and this whole expression represents an angle, which tells you where you are in the travel, a long cosine to this. What this quantity gets to be two pi, you're back where you started. That's not the distance on the t axis. It's the angle through which you go through. In other words, does number describes where you are on the cosine cycle. It doesn't tell you, it's not aiming at telling you exactly where you are on the t axis. The response function looks like 1/[sqrt(1 + (omega / k)^2)*cos(omega t - phi)].

And, I asked you on the problem set, if k goes up, in other words, if the conductivity rises, if heat can get more rapidly from the outside to the inside, for example, how does that affect the amplitude? This is the amplitude, A, and the phase lag. In other words, how does this affect the response? And now, you can see. If k goes up, this fraction is becoming smaller. That means the denominator is becoming smaller, and therefore, the amplitude is going up.

What's happening to the phase lag? Well, the phase lag looks like this: phi omega / k. If k is going up, then the size of this side is going down, and the angle is going down. Now, that part is intuitive. I would have expected everybody to get that. It's the heat gets in quickly, more quickly, then the amplitude will match more quickly. This will rise, and get fairly close to one, in fact, and there should be very little lag in the way the response follows input. But how about the other one? Okay, I give you two minutes. The other one, you will figure out yourself.