Logistic Model: Qualitative Analysis

We will approach this topic through examples. As stated in the introduction, a first order autonomous equation is one of the form

$$\dot{y} = g(y).$$

1. Simple Examples

Example 1. Natural growth or decay with constant growth-rate k:

$$\dot{y} = ky.$$

Example 2. Bank account with interest rate *not* depending on time but possibly depending upon current balance and constant savings rate:

$$\dot{y} = I(y)\, y + q.$$

2. Logistic Population Model

Example 3. The logistic population model is a simple model that takes into account the limits the environment imposes on population growth. Suppose we have a model for a population y that has a variable growth rate $k(y)$ which depends on the current population but *not on time*. That is,

$$\dot{y} = k(y) \cdot y. \tag{1}$$

Suppose that when y is small the growth rate is approximately k_0, but that there is a maximal sustainable population M. This means that as y gets near M the growth rate decreases to zero. And, if $y > M$, the growth rate becomes negative and the population declines back to the maximal sustainable population.

In the simplest version of this, the graph of $k(y)$ is a straight line with $k = k_0$ when $y = 0$ and $k = 0$ when $y = M$.

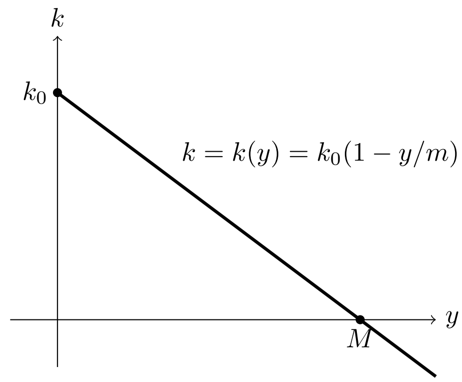

Fig. 1. Line with vertical intercept k_0 and horizontal intercept M.

The equation of this line is

$$k(y) = k_0(1 - y/M).$$

(You can check that $k(0) = k_0$, $k(M) = 0$ and $k(y) < 0$ for $y > M$.)

In this case equation (1) is known as the **Logistic Population Model** :

$$\dot{y} = k_0(1 - (y/M))y = f(y). \qquad (2)$$

This is more realistic than *natural growth* when you want to account for limits to growth. It is *nonlinear* but it *is* autonomous.

Autonomous equations are always separable and, in this case, we could compute the resulting integral using partial fractions. But we are aiming for a qualitative grasp of the solutions, which we develop in the next example.

Example 4. Give a qualitiative picture of the solutions without solving equation (2).

Solution. We start by looking for constant solutions $y(t) = y_0$. Since a constant has derivative 0, plugging this into (2) gives

$$0 = f(y_0)$$

We see that $y_0 = 0$ and $y_0 = M$ are the two points where $f(y_0) = 0$. Thus we have two constant solutions $y(t) = 0$ and $y(t) = M$. Because a system at equilibrium is unchanging, we will call these solutions **equilibrium solutions**. Since $\dot{y} = f(y) = 0$ when $y = 0$ and $y = M$ we call 0 and M the **critical points** of the DE. To summarize, the following all say the same thing:

1. $f(y_0) = 0$.

2. $y(t) = y_0$ is an equilibrium solution.
3. $y = y_0$ is a critial point.

To tie this to previous work, note the equation is separable and our constant solutions are none other than the *lost* solutions of the separable equation.

To understand the non-constant solutions we will sketch and analyze the direction field for equation (2). Clearly, each isocline, $f(y) = c$, is a horizontal straight line. For a fixed slope c, the isocline will consist of a horizontal line $y = y_0$ where $f(y_0) = c$.

As usual, first we look at the *nullcline* $f(y) = 0$. We already know the zeros of $f(y)$ are 0 and M. So the nullclines are the pair of lines $y = 0$ and $y = M$. These are exactly the constant solutions found above.

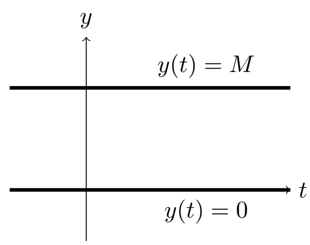

Fig. 2. The nullclines are also solution curves.

To get a clear picture of the other isoclines we will draw a graph of $f(y)$ as a function of y. It's a parabola opening downward, meeting the horizontal axis at $y = 0$ and $y = M$.

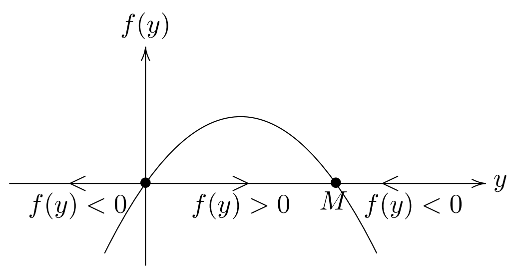

Fig. 3. The graph of $f(y)$ tells us where $\dot{y}$ is positive and negative.

The graph shows that

for $y < 0$ $\quad \dot{y} = f(y)$ is negative,
for $0 < y < M$ $\quad \dot{y} = f(y)$ is positive,
for $M < y$ $\quad \dot{y} = f(y)$ is negative.

This is indicated on the graph by the arrows on the horizontal axis. The arrow points left (towards decreasing y) where $\dot{y}$ is negative and right (towards increasing y) where $\dot{y}$ is positive. To make things clear, we also label the intervals as having $f(y)$ positive or negative.

Now we can sketch the direction field. First, we draw the nullclines and since these are horizontal lines, we don't need to sketch the direction field elements (little line segments) along them. Then we choose a horizontal line above $y = M$ and sketch the direction field elements along it. (We know they are negative because for $y > M$ we know $\dot{y} < 0$.) Similarly, we add an isocline between 0 and M and one below 0.

Finally we can sketch some solution curves:

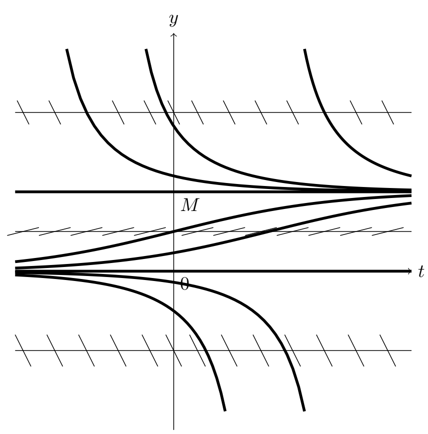

Fig. 4. Direction field and solution curves for the logistic equation (2).

1. Since the slope field is constant in the t direction any solution curve can be translated left or right and still be a solution.

2. Since the lines $y = 0$ and $y = M$ are solutions the other curves can't cross them.

3. The solutions that start just above the equilibrium solution $y = 0$ must increase. Since they can't cross the solution $y = M$ they must go asymptotically to towards it. These bounded solutions are called **logistic curves** or **S-curves**. They represent the population drifting from just above the equilibrium $y = 0$ towards the one at $y = M$.

4. If the population exceeds the M, it tends back towards it. This represents environmental pressure related to overpopulation. M is called the **carrying capacity** of the environment.

5. In a population model we would never see $y < 0$. Mathematically, the solution curves that start below $y = 0$ decrease without bound.

3. Stable and Unstable Equilibria

Notice that in the logistic model all the solution curves that start near the equilibrium $y = M$ go asymptotically towards it. (See figure 4.) We call such an equilibrium a **stable equilibrium**. Similarly, we call the equilibrium $y = 0$ an **unstable equilibrium** because all the curves that start near it move away.

4. Summary

The sketch of the solutions gives us our qualitive picture. We also defined a number of terms.

1. Autonomous equation: $\dot{y} = f(y)$.
2. Equilibrium solutions: Constant solutions $y(t) = y_0$ where $f(y_0) = 0$.
3. Critical points: The value of the equilibrium solutions, i.e., values y_0 where $f(y_0) = 0$.
4. Stable equilibrium: An equilibrium solution where all nearby solution curves tend towards it.
5. Unstable equilibrium: An equilibrium solution where all nearby solution curves tend away from it.
6. Logistic population model, logistic curves: see above.
7. Carrying capacity: The stable equilibrium in the logistic model that all (positive) populations approach asymptotically.

In later examples we will learn how to systematically make a qualitative sketch of the solution curves.