Part I Problems and Solutions

Problem 1: *For each of the following autonomous equations $dx/dt = f(x)$, obtain a qualitatitive picture of the solutions as follows:*

(i) Draw horizontally the axis of the dependent variable x, indiciating the critical points of the equation; put arrows on the axis indicating the direction of motion between the critical points and label each critical point as stable, unstable, or semi-stable. Indicate where this information comes from by including in the same picture the graph of $f(x)$, drawn with dashed lines.

(ii) Use the information in the first picture to make a second picture showing the tx-plane, with a set of typical solutions to the ODE. The sketch should show the main qualitative features (e.g., the constant solutions, asymptotic behavior of the non-constant solutions).

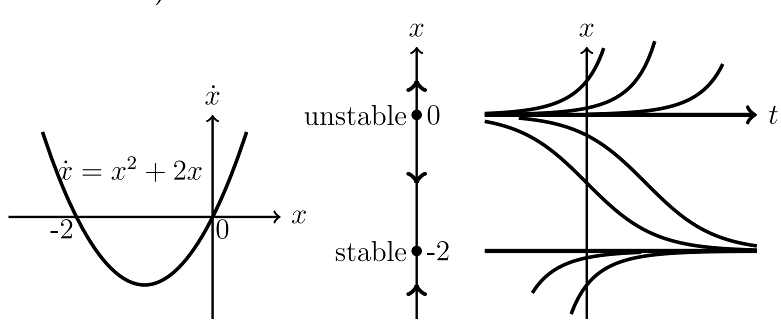
a) $x' = x^2 + 2x$

b) $x' = -(x-1)^2$

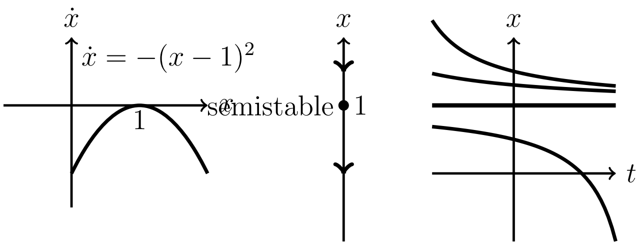
c) $x' = 2x - x^2$

d) $x' = (2-x)^3$

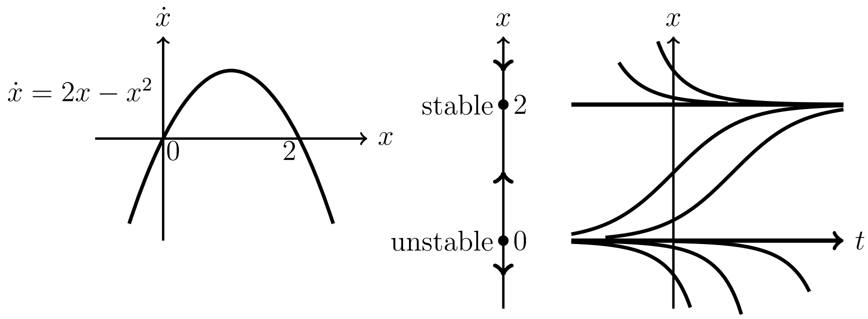
Solution: a)



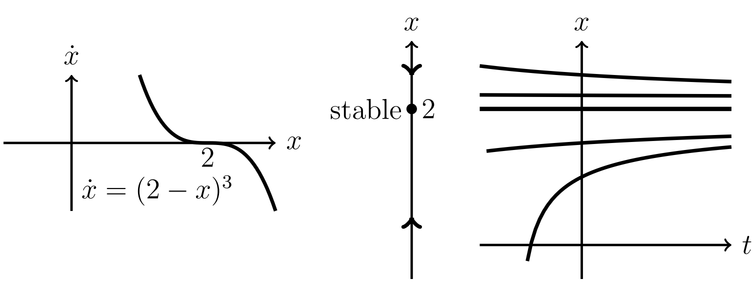
b)



c)



d)



Problem 2: *Consider the differential equation* $\dot{x} + 2x = 1$.

a) *Find the general solution three ways: (i) by separation of variables, (ii) by use of an integrating factor, (iii) by regarding the right hand side as* e^{0t} *and using the method of optimism (i.e. look for a solution of the form* Ae^{0t}*) to find a particular solution, and then adding in a transient.*

b) *This equation is also autonomous. Sketch its phase line and some solutions (including the equilibrium solution). Is the equilibrium stable, unstable, or neither?*

c) *Use Euler's method with three steps to estimate the value of the solution with initial condition* $x(0) = 0$ *at* $t = 1$.

Solution:

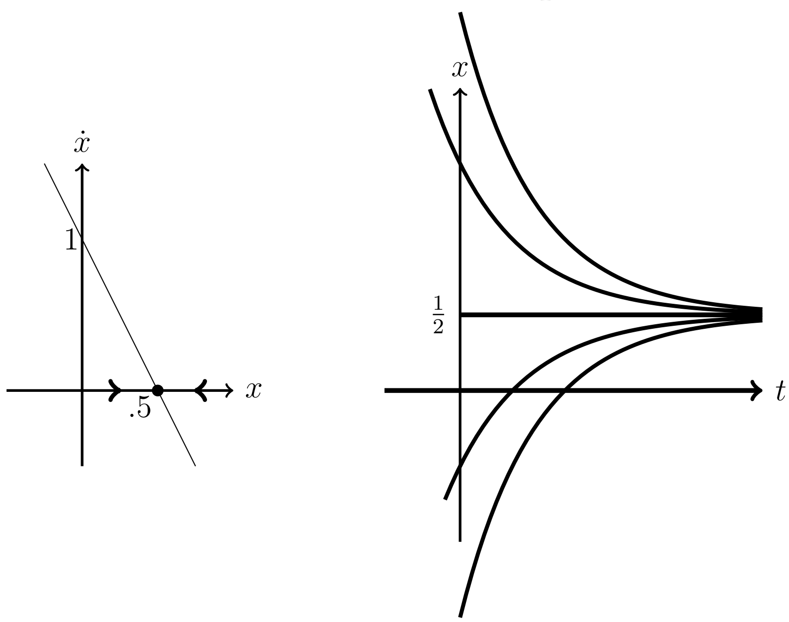
a) By separation: $\frac{dx}{dt} = 1 - 2x \rightarrow \frac{dx}{1-2x} = dt \rightarrow -\frac{1}{2}\ln(1-2x) = t + c \rightarrow 1 - 2x = e^{-2t+c} = Ce^{-2t}$ so $x = \frac{1}{2}\left(1 + Ce^{-2t}\right) = \frac{1}{2} + ce^{-2t}$ (with $c = C/2$)

By integrating factor: $\frac{dx}{dt} + 2x = 1$ IF $\rho = e^{2t} \rightarrow$
$x = e^{-2t}(c + \int 1 \cdot e^{2t} dt)$ so $x = Ce^{-2t} + e^{-2t}(\frac{1}{2}e^{2t}) \rightarrow x = \frac{1}{2} + Ce^{-2t}$.

By optimism: $x_p = Ae^{0t} = A \rightarrow \dot{x}_p = 0 \rightarrow \dot{x}_p + 2x_p = 0 + 2A = 1$ so $x_p = \frac{1}{2}$. x_h is the solution to $\dot{x} + 2x = 0 \rightarrow x_h = Ce^{-2t}$. $x = x_p + x_h$, and therefore is

$$x = \frac{1}{2} + Ce^{-2t}$$

b) $\dot{x} = 1 - 2x$. Critical point $\dot{x} = 0 \rightarrow 1 - 2x = 0 \rightarrow x = \frac{1}{2}$.



c) We use Euler's method. $\frac{dx}{dt} = f(t, x) = 1 - 2x$ with $t_0 = 0$ and $x_0 = x(t_0) = x(0) = 0$ (given). We then have:

$h = \frac{1}{3}$, $t_1 = t_0 + h = \frac{1}{3}$

$x_1 = x_0 + hf(t_0, x_0) = 0 + \frac{1}{3}f(0, 0) = 0 + \frac{1}{3}(1 - 2 \cdot 0) = \frac{1}{3}$

$t_2 = t_1 + h = \frac{2}{3}$ so $x_2 = x_1 + hf(t_1, x_1) = \frac{1}{3} + \frac{1}{3}(1 - 2 \cdot \frac{1}{3}) = \frac{4}{9}$

$t_3 = t_2 + h = 1$ so $x_3 = x_2 + hf(t_2, x_2) = \frac{4}{9} + \frac{1}{3}(1 - 2 \cdot \frac{4}{9}) = \frac{13}{27} \approx .4815$

Check against actual value:
$x(0) = 0 \rightarrow$ solution from $x(t) = \frac{1}{2}(1 - e^{-2t})$
$x(1) = \frac{1}{2}(1 - e^{-2}) \approx .4323.$
$x(t)$ is concave down. Thus, Euler approximation is too high, but seems reasonable.