

First Order Response to Sinusoidal Input

1. Introduction

We are going to solve a first order constant coefficient DE with sinusoidal input:

$$\dot{x} + kx = B \cos(\omega t). \quad (1)$$

Our strategy will be to use Euler's formula to replace $\cos(\omega t)$ by the complex exponential $e^{i\omega t}$. We call this technique **complex replacement**. We illustrate it with an example and then rework the example in the general case.

2. Illustrative Example

Solve the ODE

$$\dot{x} + 2x = 2 \cos(2t). \quad (2)$$

Solution. We will go through this example very carefully. After sufficient practice many of the steps can be done in your head.

The key is to introduce a new variable y with its own related ODE

$$\dot{y} + 2y = 2 \sin(2t). \quad (3)$$

Now we combine x and y to make a complex variable $z = x + iy$. Combining equations (1) and (3) in the same manner we get

$$\dot{z} + 2z = 2 \cos(2t) + 2i \sin(2t) = 2e^{2it}. \quad (4)$$

We note that $x = \text{Re}(z)$, so once we've found $z(t)$ we can easily find $x(t)$.

Equation (4) has exponential input and we know how to solve it: try a solution of the form $z_p(t) = Ae^{2it}$. Substituting this into the equation gives

$$\text{Left hand side: } \dot{z}_p + 2z_p = 2iAe^{2it} + 2Ae^{2it} = (2 + 2i)Ae^{2it}.$$

$$\text{Right hand side: } 2e^{2it}.$$

Equating the two sides we get

$$(2 + 2i)Ae^{2it} = 2e^{2it} \Rightarrow A = 1/(1 + i).$$

Thus, $z_p(t) = e^{2it}/(1 + i)$.

The problem asks for x which is the real part of z . We can find x using polar or Cartesian coordinates. We will do it both ways. For most students using polar coordinates is less familiar. You should therefore learn it well

because polar coordinates are easier to interpret and we are generally preferred. The sinusoidal identity can be used to convert one form to the other.

Polar Coordinates.

In polar coordinates, $1 + i = \sqrt{2} e^{i\pi/4}$. Using this in the formula for z_p :

$$z_p(t) = \frac{e^{2it}}{1+i} = \frac{e^{2it}}{\sqrt{2}e^{i\pi/4}} = \frac{e^{i(2t-\pi/4)}}{\sqrt{2}} = \frac{1}{\sqrt{2}} (\cos(2t - \pi/4) + i \sin(2t - \pi/4)).$$

Taking the real part we get

$$x_p(t) = \operatorname{Re}(z_p(t)) = \frac{1}{\sqrt{2}} \cos(2t - \pi/4).$$

Finally, as always, we add the homogeneous solution to this to get the general solution:

$$x(t) = x_p(t) + Ce^{-kt} = \frac{1}{\sqrt{2}} \cos(2t - \pi/4) + Ce^{-kt}.$$

Cartesian Coordinates.

We use the complex conjugate to handle the denominator:

$$z_p(t) = \frac{e^{2it}}{1+i} = \frac{\cos(2t) + i \sin(2t)}{1+i} \cdot \frac{1-i}{1-i} = \frac{\cos(2t) + \sin(2t) + i(\sin(2t) - \cos(2t))}{2}.$$

Taking the real part we get

$$x_p(t) = \frac{\cos(2t) + \sin(2t)}{2}.$$

Exercise. Use the sinusoidal identity to show that the two solutions given in the previous example are, in fact, identical.

3. General Case

Solve the ODE

$$\dot{x} + kx = B \cos(\omega t). \quad (5)$$

(We assume k , ω and B are all positive.)

Solution. This is really just a matter of replacing the numbers in our illustrative example by the letters k , B and ω . We will not write down as much as before. If something is unclear you can go to the corresponding part of the example above to understand it.

Do the complex replacement:

$$\dot{z} + kz = Be^{i\omega t}, \quad \text{where } \cos(\omega t) = \operatorname{Re}(e^{i\omega t}) \text{ and } x = \operatorname{Re}(z). \quad (6)$$

Equation (5) has exponential input, so we try a solution of the form $z_p(t) = Ae^{i\omega t}$. Substituting this into the equation gives

$$\text{Left hand side: } \dot{z}_p + kz_p = i\omega Ae^{i\omega t} + kAe^{i\omega t} = (k + i\omega)Ae^{i\omega t}.$$

$$\text{Right hand side: } Be^{i\omega t}.$$

Equating the two sides we get

$$(k + i\omega)Ae^{i\omega t} = Be^{i\omega t} \Rightarrow A = B/(k + i\omega).$$

Thus, $z_p(t) = Be^{i\omega t}/(k + i\omega)$. In polar coordinates

$$k + i\omega = \sqrt{k^2 + \omega^2} e^{i\phi}, \quad \text{where } \phi = \tan^{-1}(\omega/k) \text{ in the first quadrant.}$$

(Because \tan^{-1} is ambiguous, e.g. $\tan(\pi/4) = \tan(5\pi/4) = 1$, we fix the value of \tan^{-1} by saying which quadrant the complex number is in. In this case, since $k, \omega > 0$, $k + i\omega$ is in the first quadrant. Another way to do this would be to write $\phi = \operatorname{Arg}(k + i\omega)$.) Thus,

$$z_p(t) = \frac{Be^{i\omega t}}{k + i\omega} = \frac{Be^{i\omega t}}{\sqrt{k^2 + \omega^2} e^{i\phi}} = \frac{Be^{i(\omega t - \phi)}}{\sqrt{k^2 + \omega^2}}.$$

Taking the real part we get

$$x_p(t) = \frac{B}{\sqrt{k^2 + \omega^2}} \cos(\omega t - \phi).$$

Finally, as always, we add the homogeneous solution to this to get the general solution:

$$x(t) = x_p(t) + Ce^{-kt} = \frac{B}{\sqrt{k^2 + \omega^2}} \cos(\omega t - \phi) + Ce^{-kt}.$$

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