Part II Problems and Solutions

Problem 1: *[Sinusoidal input and output]*

(a) *Express* $\operatorname{Re}\left(\frac{e^{3it}}{\sqrt{3}+i}\right)$ *in the form* $a\cos(3t) + b\sin(3t)$*. Then rewrite this in the form* $A\cos(3t-\phi)$*. Now find this same answer using the following method. By finding its modulus and argument, write* $\sqrt{3}+i$ *in the form* $Ae^{i\phi}$*. Then substitute this into* $e^{3it}/(\sqrt{3}+i)$*, and use properties of the exponential function to find* B *and* ϕ *such that* $\frac{e^{3it}}{\sqrt{3}+i} = Be^{i(3t-\phi)}$*. Finally, take the real part of this new expression.*

(b) *Find a solution to the differential equation* $\dot{z} + 3z = e^{2it}$ *of the form* we^{2it}*, where* w *is some complex number. What is the general solution?*

(c) *Find a solution of* $\dot{x} + 3x = \cos(2t)$ *by relating this ODE to the one in* **(b)**. *What is the general solution?*

Solution: **(a)** $\frac{e^{3it}}{\sqrt{3}+i} = \frac{(\sqrt{3}-i)}{4}(\cos(3t) + i\sin(3t))$ has real part $\frac{\sqrt{3}}{4}\cos(3t) + \frac{1}{4}\sin(3t)$.

Form the right triangle with sides $a = \frac{\sqrt{3}}{4}$ and $b = \frac{1}{4}$. The hypotenuse is $A = 1/2$ and the angle is $\phi = \pi/6$.

$\sqrt{3}+i = 2e^{\pi i/6}$ (by essentially the same triangle), so $\frac{e^{3it}}{\sqrt{3}+i} = \frac{1}{2}e^{i(3t-\pi/6)}$: $B = \frac{1}{2}$, $\phi = \frac{\pi}{6}$, and $\operatorname{Re}(Be^{i(3t-\phi)}) = B\cos(3t-\phi)$, so you get the same answer.

(b) Substituting $z = we^{2it}$, $e^{2it} = w2ie^{2it} + 3we^{2it}$, so $1 = w(2i+3)$ or $w = \frac{1}{2i+3}$. Thus a solution of the desired form is $z_p = \frac{1}{2i+3}e^{2it}$. The general solution is $z_p + ce^{-3t}$.

(c) If $x = \operatorname{Re}(z)$, the real part of $\dot{z} + 3z = e^{2it}$ is $\dot{x} + 3x = \cos(2t)$. So we are looking for $\operatorname{Re}(z_p)$, where z_p is the answer in part (b).
In polar form, $2i + 3 = \sqrt{13}e^{i\phi}$, where $\phi = \tan^{-1}(2/3)$.
Thus,

$$z_p = \frac{1}{\sqrt{13}}e^{i(2t-\phi)}$$

We get

$$x_p = \operatorname{Re}(z_p) = \frac{1}{\sqrt{13}}\cos(2t-\phi).$$

The general solution is then $x = x_p + ce^{-3t}$.