Euler's Formula, Polar Representation

1. The Complex Plane

Complex numbers are represented geometrically by points in the plane: the number $a + ib$ is represented by the point (a, b) in Cartesian coordinates. When the points of the plane are thought of as representing complex numbers in this way, the plane is called the **complex plane**.

By switching to polar coordinates, we can write any non-zero complex number in an alternative form. Letting as usual

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

we get the **polar form** for a non-zero complex number: assuming $x + iy \neq 0$,

$$x + iy = r(\cos(\theta) + i\sin(\theta)). \tag{1}$$

When the complex number is written in polar form,

$$r = |x + iy| = \sqrt{x^2 + y^2}. \qquad \textit{(absolute value, modulus).}$$

We call θ the *polar angle* or the *argument* of $x + iy$. In symbols, one sometimes sees:

$$\theta = \arg(x + iy). \qquad \textit{(polar angle, argument).}$$

The absolute value is uniquely determined by $x + iy$ but the polar angle is not, since it can be increased by any integer multiple of 2π. (The complex number 0 has no polar angle.) To make θ unique, one can specify

$$0 \leq \theta < 2\pi. \qquad \textit{(principal value).}$$

This so-called principal value of the angle is sometimes indicated by writing $\text{Arg}(x + iy)$. For example,

$$\text{Arg}(-1) = \pi, \qquad \arg(-1) = \pm\pi, \pm 3\pi, \pm 5\pi, \cdots$$

Changing between Cartesian and polar representation of a complex number is essentially the same as changing between Cartesian and polar coordinates: the same equations are used and the same triangle appears in the plane. The figure below shows this. (You will learn what $e^{i\theta}$ means in the next section.)

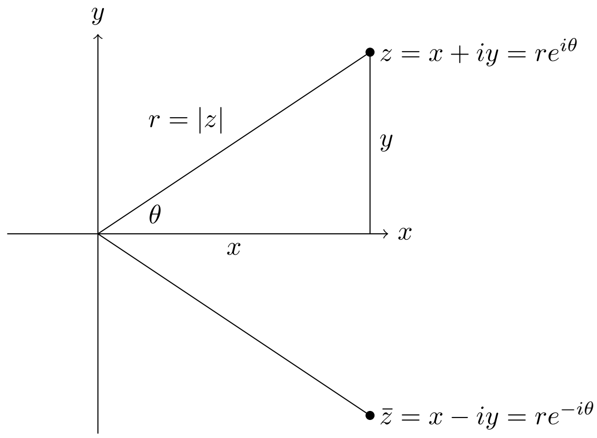

Fig 1. The complex plane.

Example 1. Give the polar form for: $-i, 1+i, 1-i, -1+i\sqrt{3}$.

Solution.

$$-i = i\sin(3\pi/2) \qquad 1+i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$$
$$-1+i\sqrt{3} = 2(\cos(2\pi/3) + i\sin(2\pi/3)) \qquad 1-i = \sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4)).$$

2. Euler's Formula

The abbreviation $\operatorname{cis}\theta$ is sometimes used for $\cos(\theta) + i\sin(\theta)$; for students of science and engineering, however, it is important to get used to the *exponential* form for this expression:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \qquad \textbf{Euler's formula.} \qquad (2)$$

Equation (2) should be regarded as the *definition* of the exponential of an imaginary power. A good justification for it is found in the infinite series:

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

If we substitute $i\theta$ for t in the series and collect the real and imaginary parts of the sum (remembering that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and so on), we get:

$$\begin{aligned} e^{i\theta} &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos(\theta) + i\sin(\theta) \end{aligned}$$

in view of the infinite series representations for $\cos(\theta)$ and $\sin(\theta)$.

Since we only know that the series expansion for e^t is valid when t is a real number, the above argument is only suggestive — it is not a proof of (2). What it shows is that Euler's formula (2) is formally compatible with the series expansions for the exponential, sine, and cosine functions.

3. Polar Representation

Using the complex exponential, the polar representation (1) is written:

$$x + iy = re^{i\theta}. \tag{3}$$

The most important reason for polar representation is that multiplication of complex numbers is particularly simple when they are written in polar form. Indeed, by using Euler's formula (2) and the trigonometric addition formulas, it is not hard to show that

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}. \tag{4}$$

This gives another justification for the definition (2) — the complex exponential follow the same exponential addition rules as the real exponential. The law (4) leads to the simple rules for multiplying and dividing complex numbers written in polar form:

Multiplication Rule

$$r_1e^{i\theta_1} \cdot r_2e^{i\theta_2} = r_1r_2e^{i(\theta_1+\theta_2)}. \tag{5}$$

To multiply two complex numbers, you multiply the absolute values and add the angles.

Reciprocal Rule

$$\frac{1}{re^{i\theta}} = \frac{1}{r}\,e^{-i\theta}; \tag{6}$$

Division Rule

$$\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}} = \frac{r_1}{r_2}\,e^{i(\theta_1-\theta_2)}. \tag{7}$$

To divide by a complex number, divide by its absolute value and subtract its angle.

The reciprocal rule (6) follows from (5), which shows that

$$\frac{1}{r}\,e^{-i\theta} \cdot re^{i\theta} = 1.$$

Using (5), we can raise $x + iy$ to a positive integer power by first using $x + iy = re^{i\theta}$:

$$(x+iy)^n = r^n e^{in\theta}; \tag{8}$$

DeMoivre's formula: The special case when $r = 1$ is called *DeMoivre's Formula*:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta). \tag{9}$$

Example 2. Express:

a) $(1+i)^6$ in Cartesian form;

b) $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i}$ in polar form.

Solution.

a) Change to polar form, use (8), then change back to Cartesian form:

$$(1+i)^6 = (\sqrt{2}e^{i\pi/4})^6 = (\sqrt{2})^6 e^{i6\pi/4} = 8e^{i3\pi/2} = -8i.$$

b) Changing to polar form, $\dfrac{1+i\sqrt{3}}{\sqrt{3}+i} = \dfrac{2e^{i\pi/3}}{2e^{i\pi/6}} = e^{i\pi/6}$, using the division rule (7).

You can check the answer to (a) by applying the binomial theorem to $(1+i)^6$ and collecting the real and imaginary parts; to (b) by doing the division in the Cartesian form then converting the answer to polar form.

3.1. Combining pure oscillations of the same frequency.

The equation which does this is widely used in physics and engineering; it can be expressed using complex numbers:

$$a\cos(\lambda t) + b\sin(\lambda t) = A\cos(\lambda t - \phi), \qquad \text{where } a + bi = Ae^{i\phi}; \tag{10}$$

in other words, $A = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}(b/a)$.
To prove (10), we have:

$$\begin{aligned} a\cos(\lambda t) + b\sin(\lambda t) &= \mathrm{Re}\left((a-bi)\cdot(\cos(\lambda t) + i\sin(\lambda t))\right) \\ &= \mathrm{Re}(Ae^{-i\phi}\cdot e^{i\lambda t}) \\ &= \mathrm{Re}(Ae^{i(\lambda t-\phi)}) = A\cos(\lambda t - \phi). \end{aligned}$$