

Part II Problems and Solutions

Problem 1: [Complex numbers, roots of unity]

(a) Make a table with three columns. Each row will contain three representations of a complex number z : the “rectangular” expression $z = a + bi$ (with a and b real); the “polar” expression $|z|, \text{Arg}(z)$; and a little picture of the complex plane with the complex number marked on it. There are five rows, containing, in one column or another, the following complex numbers:

(i) $1 - i$

(ii) z such that $|z| = 2$ and $\text{Arg}(z) = \pi/6$

(iii) A square root of i with negative real and imaginary parts

(iv) A sixth root of 1 with argument θ such that $0 < \theta < \pi/2$

(v) $((1 + i)/\sqrt{2})^{-13}$.

(b) Find the complex roots of the following equations: $z^4 + 4 = 0$; $z^2 + 2z + 2 = 0$.

Solution: (a) The rectangular expression gives the coordinates for the little pictures. Any angle may be altered by adding a multiple of 2π . The table below does not have a column for the pictures – you will have to supply those. The last column is for problem 2(a)

$1 - i$	$\sqrt{2}, -\pi/4$	$\sqrt{2}e^{-\pi i/4}$
$\sqrt{3} + i$	$2, \pi/6$	$2e^{\pi i/6}$
$(-1 - i)/\sqrt{2}$	$1, 5\pi/4$	$e^{5\pi i/4}$
$(1 + \sqrt{3}i)/2$	$1, \pi/3$	$e^{\pi i/3}$
$(-1 + i)/\sqrt{2}$	$1, 3\pi/4$	$e^{3\pi i/4}$

(b) (i) $\pm 1 \pm i$; or $\sqrt{2}e^{k\pi i/4}$ where $k = 1, 3, 5, 7$. (ii) $-1 \pm i$.

Problem 2: [Complex exponentials, Sinusoids]

(a) Add a fourth column to table you made in 1(a) by giving the exponential representation $z = Ae^{i\theta}$ (with A and θ real).

(b) Find all complex numbers $z = a + bi$ such that $e^z = -2$.

(c) Find an expression for $\cos(4t)$ in terms of sums of powers of $\sin t$ and $\cos t$ by using $(e^{it})^4 = e^{4it}$ and Euler’s formula.

(d) The Mathlet Complex Exponential will probably be useful in understanding the rest of this problem. Open it and explore its functionalities. The Help button lists most of them. Notice that in the left window, the real part a ranges between -1 and 1 , while the imaginary part b ranges from -8 to 8 . You use the left-hand window to pick out a complex

number $a + bi$. When you do, a portion of the line through it and zero is drawn. This line is parametrized by $(a + bi)t$. At the same time, the curve parametrized by the complex-valued function $e^{(a+bi)t}$ is drawn on the right window.

For each of the following functions $f(t)$, make a sketch of the graph (in a convenient range), find a value of $w = a + bi$ such that $\operatorname{Re}(e^{wt}) = f(t)$, and sketch the trajectory of the complex valued function e^{wt} (i.e. its set of values).

(i) $f(t) = \cos(2\pi t)$

(ii) $f(t) = e^{-t}$

(iii) $f(t) = e^{-t} \cos(2\pi t)$.

(iv) The constant function with value 1.

Solution: (a) The solution in the right hand column of the table above.

(b) $e^{a+bi} = e^a e^{bi}$ so $|e^{a+bi}| = |e^a| |e^{bi}| = e^a$. Since $|-2| = 2$, $a = \ln 2$. $\operatorname{Arg}(e^{a+bi}) = b$ up to adding multiples of 2π . $\operatorname{Arg}(-1) = \pi$, so b is any odd multiple of π . Answer: $\ln 2 + b\pi i$, $b = \pm 1, \pm 3, \dots$

(c) $\cos(4t) = \operatorname{Re}(e^{4it}) = \operatorname{Re}((e^{it})^4) = \operatorname{Re}((\cos t + i \sin t)^4)$. By the binomial theorem, $(a + bi)^4 = a^4 + 4a^3bi - 6a^2b^2 - 4ab^3i + b^4$, so we find $\cos(4t) = \cos^4 t - 6 \cos^2 t \sin^2 t + \sin^4 t$.

(d) (i) $w = 2\pi i$. The trajectory is the unit circle.

(ii) $w = -1$. The trajectory is the positive real axis.

(iii) $w = -1 + 2\pi i$. The trajectory is a spiral, spiralling in towards the origin in a counter-clockwise direction and passing through 1.

(iv) $w = 0$. The trajectory is the single point 1.

MIT OpenCourseWare
<http://ocw.mit.edu>

18.03SC Differential Equations
Fall 2011

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.