Part I Problems and Solutions

Problem 1: *Find the general solution by separation of variables:*

$$\frac{dy}{dx} = 2 - y, \qquad y(0) = 0$$

Solution:

$$\frac{dy}{2-y} = dx \rightarrow \int \frac{dy}{2-y} = \int dx \rightarrow$$

$$-\ln|2-y| = x + c \rightarrow |2-y| = Ce^{-x} \quad (\text{with } C = e^c) \rightarrow$$

$$y = 2 - Ce^{-x} \qquad (\text{with } C \text{ any number})$$

IC: $y(0) = 2 - C = 0 \rightarrow C = 2 \rightarrow y = 2(1 - e^{-x})$

Problem 2: *Find the general solution by separation of variables:*

$$\frac{dy}{dx} = \frac{(y-1)^2}{(x+1)^2}$$

Solution:

$$\frac{dy}{(y-1)^2} = \frac{dx}{(x+1)^2} \rightarrow \int \frac{dy}{(y-1)^2} = \int \frac{dx}{(x+1)^2} \rightarrow$$

$$-\frac{1}{y-1} = -\frac{1}{x+1} + c$$

Extra: solve for y as a function of x:

Answer: $y = 1 + \frac{x+1}{1-c(x+1)}$

Problem 3: *The rate of change of a certain population is proportional to the square root of its size. Model this situation with a differential equation.*

Solution: Let $P = P(t)$ be the size of the population as a function of time t. Then

$$\frac{dP}{dt} = k\sqrt{P}$$

where $k > 0$ is the constant of proportionality.

Problem 4:

The rate of change of the velocity of an object is proportional to the square of the velocity. Model this situation with a differential equation.

Solution:

$$\frac{dv}{dt} = kv^2$$

with $k > 0$, constant.

Problem 5:

In a population of fixed size S, the rate of change of the number N of persons who have heard a rumor is proportional to the number of those who have not yet heard it. Model this situation with a differential equation.

Solution:

$$\frac{dN}{dt} = k(S - N)$$

with $k > 0$, constant.

Problem 6: *The amount of a certain medicine in the bloodstream decays exponentially with a half-life of 5 hours. In order to keep a patient safe during a one-hour procedure, there needs to be at least 50 mg of medicine per kg of body weight. How much medicine should be administered to a 60kg patient at the start of the procedure?*

Solution: Let $x(t)$ be the amount of the medicine in mg present in the bloodstream at time t in hours.

Given information: $x(t) = x_0 e^{-kt}$ with $k = \frac{\ln 2}{5}\left(\frac{1}{\text{hr}}\right)$, since half-life is given as 5 hours.

Since $x(t) = x_0 e^{-kt}$ is decreasing, $x(t) \leq x(1)$ for $t \geq 1$ hrs. So the patient will be (just) safe if $x(1) = x_0 e^{-k \cdot 1} = 50 \cdot 60 = 3000$ mg $= 3$ g where $k = \frac{\ln 2}{5}$. Thus, $x_0 = x(1) \cdot e^k = e^{(\ln 2)/5} \cdot 3000 \approx 3446$ mg (or about 3.446 g).

Problem 7: *Early one morning it starts to snow. At 7AM a snowplow sets off to clear the road. By 8AM, it has gone 2 miles. It takes an additional 2 hours for the plow to go another 2 miles. Let $t = 0$ when it begins to snow, let x denote the distance traveled by the plow at time t. Assuming the snowplow clears snow at a constant rate in cubic meters/hour:*

a) Find the DE modeling the value of x.

b) When did it start snowing?

Solution: a) One approach: let k_1 be the rate (in height/hour) of snowfall and k_2 be the rate of snow clearance.

The height of snow is $k_1 t \to \Delta x \cdot k_1 t \approx k_2 \Delta t \to \frac{\Delta x}{\Delta t} \approx \frac{k_2}{k_1 t}$

This is then $\frac{dx}{dt} = \frac{k}{t}$, where k is a constant.

b) Solving by separation of variables, $x = k \ln t + C$.

Let $t = t_1$ at 7AM, so $t = t_1 + 1$ at 8AM and $t = t_1 + 3$ at 10AM.

2 miles between 7 and 8AM $\to 2 = x(t_1 + 1) - x(t_1) = k \ln\left((t_1 + 1)/t_1\right)$

4 miles between 7 and 10AM $\to 4 = x(t_1 + 3) - x(t_1) = k \ln\left((t_1 + 3)/t_1\right)$

Thus, $\ln\left(\frac{t_1+3}{t_1}\right) = 2\ln\left(\frac{t_1+1}{t_1}\right) \to \frac{t_1+3}{t_1} = \left(\frac{t_1+1}{t_1}\right)^2$

After a little algebra, $t_1 = 1$, so the snow started at 6AM

Problem 8: *A tank holds 100 liters of water which contains 25 grams of salt initially. Pure water then flows into the tank, and salt water flows out of the tank, both at 5 liters/minute. The mixture is kept uniform at all times by stirring.*

a) Write down the DE with IC for this situation.

b) How long will it take until only 1 gram of salt remains in the tank?

Solution: Let $x = x(t)$ be the amount of salt in the tank in grams, with t the time in minutes.

a) DE: net rate of change of salt. $\frac{dx}{dt}$ = salt rate in - salt rate out = $0 - 5 \cdot \frac{x}{100}$.

DE: $\frac{dx}{dt} = -.05x$, IC $x(0) = 25$

b) Solution to DE: $x(t) = Ce^{-.05t}$. IC $x(0) = 25 = c$, so $x(t) = 25e^{-.05t}$.

$x(t) = 1$ when $25e^{-.05t} = 1 \to t = \frac{\ln 25}{.05} \approx 64.38$ min