18.034 Honors Differential Equations
Spring 2009

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1. Suppose A is an $n \times n$ matrix and $\mathbf{y}_1(t), \mathbf{y}_2(t), \ldots \mathbf{y}_n(t)$ are solutions to $\mathbf{y}' = A\mathbf{y}$. Show that the set if $\{\mathbf{y}_i(t_0)\}_{i=1}^n$ is linearly independent at some time t_0, then to any other solution $\mathbf{y}(t)$ there correspond constants c_i so that $\mathbf{y}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t) + \ldots + c_n\mathbf{y}_n(t)$ (i.e., the set $\{\mathbf{y}_i(t)\}_{i=1}^n$ constitutes a basis of solutions).

2. Let A be an $n \times n$ matrix.

   (a) Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A corresponding to the eigenvalues λ_1 and λ_2, respectively. If $\lambda_1 \neq \lambda_2$, show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

   (b) Assume now that $n = 2$. If $p_A(\lambda) = (\lambda - \lambda_1)^2$, show that *either* $A = \lambda_1 I$, *or* there is a unique eigenvector $\mathbf{v}_1$ associated to λ_1 and a vector $\mathbf{v}_2$ satisfying $(A - \lambda_1)\mathbf{v}_2 = \mathbf{v}_1$.

   (c) For A as in the latter alternative in (2), show that the general solution to
   $$\frac{d}{dt}\mathbf{y} = A\mathbf{y}$$
   is given by $\mathbf{y} = e^{\lambda_1 t}(c_1 t + c_2)\mathbf{v}_1 + c_1 e^{\lambda_1 t}\mathbf{v}_2$.

3. For the system
   $$y_1' = 3y_1 + 2y_2, \quad y_2' = -2y_1 - y_2,$$
   find the unique fundamental matrix $U(t)$ satisfying $U(0) = I$.

4. Under what conditions on the trace and determinant of the 2×2 matrix A will *all* solutions to the equation $\mathbf{y}' = A\mathbf{y}$ satisfy $\lim_{t\to\infty} |\mathbf{y}(t)| = 0$?