18.034 Honors Differential Equations
Spring 2009

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1. Define an inner or "dot" product on $C[a, b]$ by

$$\langle u, v\rangle = \int_a^b u(x)v(x)\,dx.$$

   (Thus $\langle\cdot,\cdot\rangle$ is a positive definite symmetric bilinear form on $C[a, b]$.) Suppose $L[u] = u'' + pu' + qu$ is a given differential operator, and $M[u]$ is its adjoint. Show that $\langle L[u], v\rangle = \langle u, M[v]\rangle$ for all u, $v \in C^2[a, b]$ provided $u(a) = u(b) = v(a) = v(b) = 0$.

2. Consider the equation $y' = F(x, y)$, where $F : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and satisfies the *(two-sided) Lipschitz condition* in y

$$|F(x, y_2) - F(x, y_1)| \le C|y_2 - y_1|.$$

   Prove that if u and v are two solutions to the equation with $u(x_0) = v(x_0)$, then $u \equiv v$

3. Suppose w is a solution to $e^{\cos x}w'' - x^2w + x^3 = 0$ with $w(0) = 0$. Let $y = w - x$ and show that y cannot have either a positive maximum or negative minimum. Noting that w'' has the sign of $w - x$, sketch the graphs of a few solution curves with varying values of $w'(0)$.

4. (Birkhoff-Rota, p. 57, #2) Reduce the following ODE to self-adjoint form.

   (a) $(1 - x^2)u'' - xu' + \lambda u = 0$.

   (b) $x^2u'' + xu' + u = 0$.