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LECTURE 30. PHASE PLANES II

Isoclines. We continue study the behavior of the trajectories of

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{30.1}$$

By forming the quotient of the two equations, we obtain

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{cx+dy}{ax+by} = \frac{c+d(y/x)}{a+b(y/x)}, \tag{30.2}$$

provided that $x \neq 0$ and $ax + by \neq 0$.

It is readily seen from (30.2) that if y/x is constant then dy/dx is constant. That means, at all points on $y = mx$, where m is a constant, the orbits passing through the points have the same slope. Such a line $y = mx$ is called an *isocline* of (30.1). The word "iso" means "same".

On the x-axis ($y = 0$), the slope of the orbits of (30.1) is c/a. Therefore, the x-axis is an isocline of (30.1). Similarly, the y-axis is an isocline of (30.1). Indeed, on the y-axis ($x = 0$), the slope of the orbits is d/b. Other useful isoclines are the line $ax + by = 0$, on which the slope is ∞, that is, the orbit is vertical, and the line $cx + dy = 0$, on which the orbits are horizontal.

An isocline is not a solution of (30.1). But, if

$$m = \frac{c+dm}{a+bm},$$

then $y = mx$ is a solution of (30.1). The condition is equivalent to that m is a root of

$$bm^2 + (a-d)m + c = 0. \tag{30.3}$$

Assume $b \neq 0$. (If $b = 0$ then $x = 0$ is an orbit which corresponds to $m = \infty$. See the exercise below.) The discriminant $\triangle = (a-d)^2 - 4bc$ of (30.3) is the discriminant of $p_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc)$. Hence, there are two straight-line orbits $y = mx$ in case of a node or a saddle ($\triangle > 0$) but no straight-line orbits in case of a focus ($\triangle < 0$).

Exercise. If m solves $bm^2 + (a-d)m + c = 0$, then show that $a + bm$ solves the characteristic equation $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$.

If $y = mx$ is a solution of (30.1) then $x' = x(a + bm)$, and hence, $x = ce^{(a+bm)t}$. Therefore, the sign of $a + bm$ determines whether the orbit $y = mx$ moves toward the origin or away from the origin. If $a + bm < 0$ then the orbit $y = mx$ is oriented toward the origin and if $a + mb > 0$ then it is away from the origin.

Exercise. For the system

$$x' = ax, \qquad y' = cx + dy$$

with $a \neq d$, show that

$$x = c_1 e^{at}, \qquad cx + (d-a)y = c_2 e^{dt}.$$

Hence,

$$u = x, \qquad v = cx + (d-a)y$$

satisfy $|u|^d = k|v|^a$. Sketch the trajectories.

Summary. We summarize our discussion so far in the diagram below.

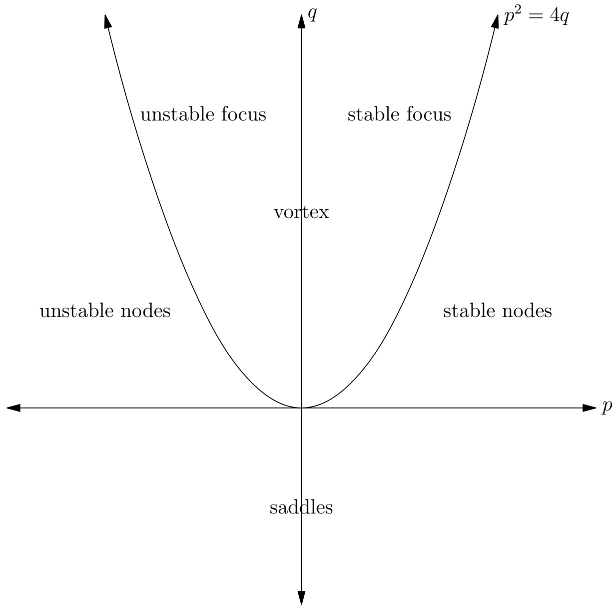

Figure 30.1.

Degenerate cases. We now consider the limiting or degenerate cases which fall outside of the six regions in Figure 30.1.

Degenerate nodes. Suppose $p^2 = 4q$ and $p \neq 0$, and suppose it does not happen that $a = d$ and $b = c = 0$. Then, $p(\lambda)$ has only one (double) root, $-p/2$. This case may be thought of two roots with the same sign, but it happens that the two roots coincide. Hence, we expect that the behavior of the orbits are similar to the case of a node. The difference, though, is that $m = \dfrac{c + dm}{a + bm}$ has one solution and hence the two straight-line orbits coincide. Thus, the orbits are tangent to a line at the origin and almost parallel to the same line at the distant points. In this case, the origin is called a *degenerate node*.

If $p > 0$, then the origin is a stable (degenerate) node, and if $p < 0$, it is a unstable (degenerate) node.

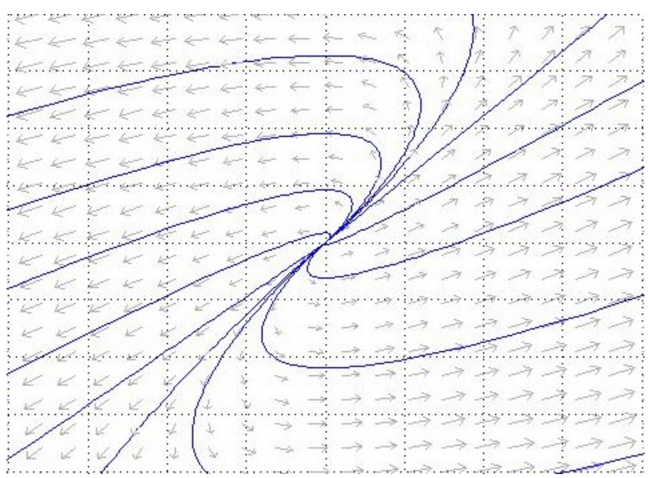

Figure 30.2. A degenerate node.

Star points. Suppose $p^2 = 4q$, and suppose $a = d$ and $b = c = 0$. Then, (30.1) reduces to

$$x' = ax, \qquad y' = ay.$$

The solution is given as

$$x(t) = c_1 e^{at}, \qquad y(t) = c_2 e^{at},$$

yielding y/x is constant. Hence, the orbits are radial lines throught the origin. In this case the origin is called *a star point* or a *singular node*. If $p > 0$ ($a > 0$), the origin is a stable star point and if $p < 0$, then it is unstable.

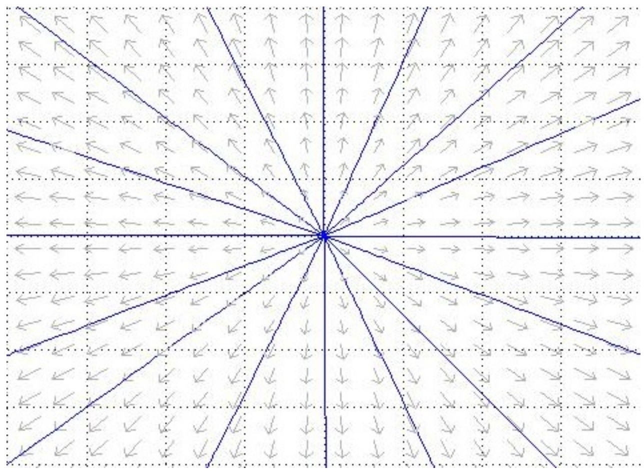

Figure 30.3. A star point.

Vanishing determinant. Next, we consider $q = 0$, that is, $ad - bc = 0$. It then follows that $cx' - ay' = 0$, and hence $cx - ay$ is constant. It gives a family of parallel lines, unless $a = c = 0$. Similarly, $dx - by$ is constant, and it gives a family of parallel lines unless $b = d = 0$.

Zero coefficient. Finally, if $p = q = 0$, then $a = b = c = d = 0$. In this case, every solution of (30.1) is a critical point. This case is uninteresting.