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LECTURE 29. PHASE PLANES I

In the theory of $u'' = f(t, u, u')$, linear or nonlinear, $u(t)$ often describes the position at time t of a point moving on the u axis and $v(t) = u'(t)$ describes the velocity at time t of the point. Together, $(u(t), v(t))$ determines the status of the system.

The behavior of the system can be described by the locus of the point $(u(t), v(t))$ in the (u, v) -plane. The (u, v) -plane associated to a differential equation this way is called the *phase plane*. The (parameterized) solution curve $(u(t), v(t))$ is called a *trajectory*, and its image is called an *orbit* or a *trace*. The main difference between a trajectory and an orbit is that the trajectory is equipped with the parameter t , which gives the *orientation* of the curve.

In this note and the following, we study the qualitative behavior of the second-order (linear) differential equations or, more generally, the system of two (linear) differential equations by plotting the trajectories in the phase plane.

In the simplest case,

$$(29.1) \quad u'' + pu' + qu = 0, \quad p, q \text{ are constants,}$$

we have $u' = v$ and $v' = -pu' - qu$. Hence, (u, v) satisfies

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

We consider, more generally, a plane autonomous system of linear equations

$$(29.2) \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The origin $(x(t), y(t)) \equiv 0$ is always a solution of (29.2), called a *critical point*, a stationary solution, or an equilibrium of (29.2). If A is non-singular, that is $|A| \neq 0$, then $(0, 0)$ is the only critical point of (29.2). The case $|A| = 0$ is called degenerate.

Let

$$p(\lambda) = \lambda^2 - (\text{tr } A)\lambda + \det A =: \lambda^2 + p\lambda + q$$

be the characteristic polynomial of A . The secular equation of (29.2) is (29.1), and it establishes a connection between a linear plane autonomous system (29.2) and its associated second-order linear differential equation.

We say (29.2) is *asymptotically stable* if all solutions tend to zero as $t \rightarrow \infty$, *stable* if all solutions remain bounded as $t \rightarrow \infty$, and *unstable* otherwise.

Linear equivalence. Two first-order linear systems

$$\begin{pmatrix} x \\ y \end{pmatrix}' = A \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix}' = B \begin{pmatrix} u \\ v \end{pmatrix}$$

are said to be *linearly equivalent* if $B = KAK^{-1}$ for a non-singular matrix K .

We write $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, and $\vec{u} = K\vec{x}$, $\vec{x} = K^{-1}\vec{u}$. Then, $\vec{x}' = A\vec{x}$ is transformed under the change of basis associated to K into

$$\vec{u}' = K\vec{x}' = KA\vec{x} = (KAK^{-1})\vec{u}$$

That is, linearly equivalent systems are associated with *similar* matrices A and KAK^{-1} . Therefore, we may reduce a linear autonomous system (29.2) to a standard simplified “canonical” form under linear equivalence.

Theorem 29.1. *Unless $\Delta = (\text{tr}A)^2 - 4 \det A = 0$, the system (29.2) is equivalent to another system if and only if they have the same secular equation.*

Lemma 29.2. *Two linearly equivalent systems have the same secular equation.*

Proof. Let’s say two equivalent systems are described by the matrices A and B , and let $B = KAK^{-1}$. We compute

$$\begin{aligned} p_B(\lambda) &= |B - \lambda I| = |KAK^{-1} - \lambda I| \\ &= |KAK^{-1} - K(\lambda I)K^{-1}| = |K(A - \lambda I)K^{-1}| = |K| |A - \lambda I| |K|^{-1} \\ &= p_A(\lambda). \end{aligned}$$

That is, their secular equations have the same characteristic polynomial. This completes the proof. \square

Lemma 29.3. *Unless $a = d$ and $b = c = 0$, the system (29.2) is linearly equivalent to*

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

where $p = -(a + d)$ and $q = ad - bc$.

Proof. If $b \neq 0$ we try the transformation $u = x$ and $v = ax + by$. That is,

$$\begin{pmatrix} u \\ v \end{pmatrix} = K \begin{pmatrix} x \\ y \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}.$$

Then,

$$\begin{aligned} u' &= x' = ax + by = v, \\ v' &= ax' + by' = av + b(cx + dy) = (a + d)v - (ad - bc)u. \end{aligned}$$

The second equation uses that $u = x$ satisfies the secular equation $u'' + pu' + qu = 0$. This shows that $KAK^{-1} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}$.

If $c \neq 0$, similarly, we try the transformation $u = y, v = cx + dy$.

Finally, if $a \neq d$ and $b = c = 0$, we try $u = x + y, v = ax + by$. \square

In the exceptional case, when $a = d$ and $b = c = 0$, the system (29.2) reduces to

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

and x and y satisfies the secular equation $u'' - 2au' + a^2u = 0$. The system

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

has the same secular equation, but they are *not* linearly equivalent.

Classification. We use the results in the previous subsections to study the behavior of (29.2). Let

$$p(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc) =: \lambda^2 + p\lambda + q,$$

and

$$\Delta = p^2 - 4q = (a - d)^2 + 4bc.$$

Focal points. If $\Delta < 0$, then $p(\lambda)$ has two distinct complex roots, say $\alpha \pm i\beta$, where $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$. In this case, we choose the canonical form

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

which has the same secular equation. The canonical form is easy to solve and it has solutions

$$u(t) = c_1 e^{\alpha t} \cos \beta t, \quad v(t) = c_2 e^{\alpha t} \sin \beta t.$$

The solution of the canonical system satisfies

$$(29.3) \quad (u/c_1)^2 + (v/c_2)^2 = e^{2\alpha t}.$$

If, in addition $\alpha \neq 0$, then the locus $(u(t), v(t))$ represents a spiral. In fact, when t increases by $2\pi/\beta$, the point (u, v) returns to the same radial line from which it started, but the exponential factor multiplies its distance to the origin by $e^{2\pi\alpha/\beta}$. The critical point thus is called a *focus*.

When $p = -2\alpha > 0$, the curve $(u(t), v(t))$ spirals toward to the origin, and we say the origin is a *stable focus*. When $p < 0$, the curve diverges from the origin and it is an *unstable focus*.

Vortex points. If $\alpha = 0$, or equivalently, $p = 0$, but $\beta \neq 0$ in the above situation, then (29.3) gives that the locus $(u(t), v(t))$ is an ellipse. In this case we say that the origin is a *vortex point*. In this case, each orbit is bounded, and hence the origin is *neutrally stable*.

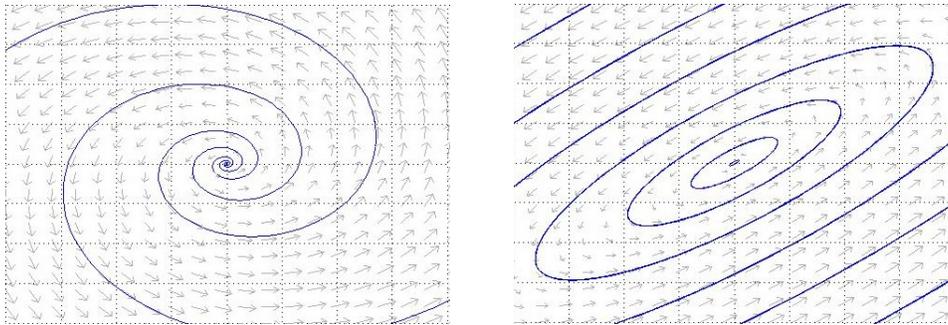


Figure 29.1. A focus (left) and a vortex (right).

Nodal points. If $\Delta > 0$ and $q > 0$, then $p(\lambda)$ has two distinct real roots of the same sign, say λ_1 and λ_2 . Let $0 < |\lambda_1| < |\lambda_2|$. We choose the canonical form to be

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

It is immediate that the canonical system has the solutions

$$u(t) = c_1 e^{\lambda_1 t}, \quad v(t) = c_2 e^{\lambda_2 t}.$$

By eliminating t from the above, we obtain

$$v = k u^m, \quad m = \lambda_2/\lambda_1 > 0,$$

and these curves in the (u, v) -plane resemble the parabolas $v = k u^2$. In this case, we say the origin is a *nodal point*.

If $p > 0$, that is, if λ_1 and λ_2 are negative, then it gives a stable node, and if $p < 0$ then it gives an unstable node.

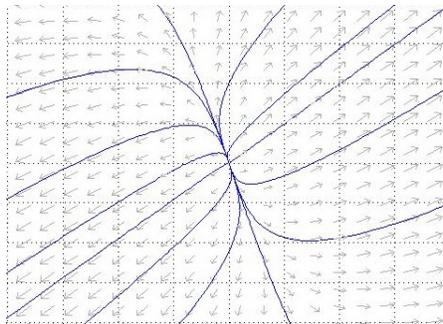


Figure 29.2. A node.

Saddle points. If $\Delta > 0$ and $q < 0$, then $p(\lambda)$ has two real roots of opposite sign, say λ_1 and λ_2 . We choose the same canonical form

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

as in the case of nodes. In this case, the solutions of this canonical system lead to

$$u^m v = k, \quad m = -\lambda_2/\lambda_1 > 0.$$

They resemble the hyperbolas $uv = k$. We say the origin is a *saddle*. A saddle point is always unstable, as in one principal direction the solution curves move toward the origin but in the other principal direction they move away from the origin.

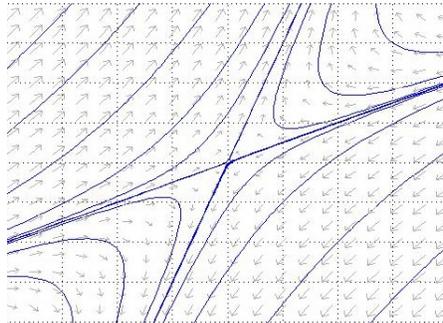


Figure 29.3. A saddle.

Example 29.4. Check that for the two systems

$$\begin{cases} x' = 6x + y, \\ y' = 4x + 3y, \end{cases} \quad \begin{cases} x' = 6x + y, \\ y' = 4x + 3y \end{cases}$$

the origin is an unstable node and a saddle, respectively. The roots of their characteristic polynomials are 2, 7 in the first case and $-2, 3$ in the second. Thus, the solutions may be written as

$$\begin{cases} x = ke^{2t} \\ y = -4x \end{cases} \quad \text{and} \quad \begin{cases} x = ke^{7t} \\ y = x \end{cases} \quad \begin{cases} x = ke^{-2t} \\ 2y = x \end{cases} \quad \text{and} \quad \begin{cases} x = ke^{3t} \\ y = 3x, \end{cases}$$

where k is constant.

In the first case, $|x|$ and $|y|$ increase with t since the exponents are positive and the lines are directed away from the origin. As $t \rightarrow -\infty$ the term e^{2t} is much larger than e^{7t} and hence the orbits are tangent at the origin to the line $y = -4x$ associated to $\lambda = 2$. As $t \rightarrow \infty$, on the other

hand, e^{7t} is much larger than e^{2t} and the orbits behave like $y = x$ at distinct points. See Figure 29.2.

In the second case the line $2y = x$ is associated with the negative exponent -2 and is directed toward the origin, while the other line is directed away from the origin. See Figure 29.3.

The additional information helps to sketch the trajectories more precisely.