18.034 Honors Differential Equations
Spring 2009

LECTURE 27. COMPLEX SOLUTIONS AND THE FUNDAMENTAL MATRIX

Complex eigenvalues. We continue studying

$$\vec{y}\,' = A\vec{y}, \tag{27.1}$$

where $A = (a_{ij})$ is a constant $n \times n$ matrix. In this subsection, further, A is a real matrix. When A has a complex eigenvalue, it yields a complex solution of (27.1). The following *principle of equating real parts* then allows us to construct real solutions of (27.1) from the complex solution.

Lemma 27.1. *If $\vec{y}(t) = \vec{\alpha}(t) + i\vec{\beta}(t)$, where $\vec{\alpha}(t)$ and $\vec{\beta}(t)$ are real vector-valued functions, is a complex solution of* (27.1)*, then both $\vec{\alpha}(t)$ and $\vec{\beta}(t)$ are real solutions of* (27.1)*.*

The proof is nearly the same as that for the scalar equation, and it is omitted.

Exercise. If a real matrix A has an eigenvalue λ with an eigenvector $\vec{v}$, then show that A also has an eigenvalue $\bar{\lambda}$ with an eigenvector $\bar{\vec{v}}$.

Example 27.2. We continue studying $A = \begin{pmatrix} 1 & 1 \\ -4 & 1 \end{pmatrix}$. Recall that $p_A(\lambda) = \begin{vmatrix} 1-\lambda & 1 \\ -4 & 1-\lambda \end{vmatrix}$ has two *complex* eigenvalues $1 \pm 2i$.

If $\lambda = 1 + 2i$, then $A - \lambda I = \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix}$ has an eigenvector $\begin{pmatrix} 1 \\ 2i \end{pmatrix}$. The result of the above exercise then ensures that $\begin{pmatrix} 1 \\ -2i \end{pmatrix}$ is an eigenvector of the eigenvalue $\lambda = 1 - 2i$.

In order to find real solutions of (27.1), we write

$$e^{(1+2i)t}\begin{pmatrix} 1 \\ 2i \end{pmatrix} = e^t \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + ie^t \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}.$$

The above lemma then asserts that $e^t \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix}$ and $e^t \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$ are real solutions of (27.1). Moreover, they are linearly independent. Therefore, the general real solution of (27.1) is

$$c_1 e^t \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + c_2 e^t \begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}.$$

The fundamental matrix. The linear operator $T\vec{y} := \vec{y}\,' - A\vec{y}$ has a natural extension from vectors to matrices. For example, when $n = 2$, let

$$T\begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \qquad T\begin{pmatrix} y_{12} \\ y_{22} \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}.$$

Then,

$$T\begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix}.$$

In general, if A is an $n \times n$ matrix and $Y = (y_1 \ \cdots \ y_n)$ is an $n \times n$ matrix, whose j-th column is y_j, then

$$TY = T(y_1 \ \cdots \ y_n) = (Ty_1 \ \cdots \ Ty_n).$$

In this sense, $\vec{y}\,' = A\vec{y}$ extends to $Y' = AY$.

Exercise. Show that

$$T(U+V) = TU + TV, \qquad T(UC) = (TU)C, \qquad T(U\vec{c}) = (TU)\vec{c},$$

where U, V are $n \times n$ matrix-valued functions, C is an $n \times n$ matrix, and $\vec{c}$ is a column vector.

That means, T is a linear operator defined on the class of matrix-valued functions Y differentiable on an interval I. The following existence and uniqueness result is standard.

Existence and Uniqueness result. . If $A(t)$ and $F(t)$ are continuous and bounded (matrix-valued functions) on an interval $t_0 \in I$, then for any matrix Y_0 then initial value problem

$$Y' = A(t)Y + F(t), \qquad Y(t_0) = Y_0$$

has a unique solution on $t \in I$.

Working assumption. $A(t)$, $F(t)$, and $f(t)$ are always continuous and bounded on an interval $t \in I$.

Definition 27.3. A *fundamental matrix* of $TY = 0$ is a solution $U(t)$ for which $|U(t_0)| \neq 0$ at some point t_0.

We note that the condition $|U(t_0)| \neq 0$ implies that $|U(t)| \neq 0$ for all $t \in I$. We use this fact to derive solution formulas.

As an application of $U(t)$, we obtain solution formulas for the initial value problem

$$\vec{y}\,' = A(t)\vec{y} + \vec{f}(t), \qquad \vec{y}(t_0) = \vec{y}_0.$$

Let $U(t)$ be a fundamental matrix of $Y' = A(t)Y$. In the homogeneous case of $\vec{f}(t) = 0$, let $\vec{y}(t) = U(t)\vec{c}$, where $\vec{c}$ is an arbitrary column vector. Then,

$$\vec{y}\,' = U'\vec{c} = (A(t)U)\vec{c} = A(t)(U\vec{c}) = A(t)\vec{y},$$

that is, y is a solution of the homogeneous system. The initial condition then determines $\vec{c}$ and $\vec{c} = U^{-1}(t_0)\vec{y}_0$.

Next, for a general $\vec{f}(t)$, we use the variation of parameters by seting $\vec{y}(t) = U(t)\vec{v}(t)$, where $\vec{v}$ is a vector-valued function. Then,

$$\vec{y}\,' = (U\vec{v})' = U'\vec{v} + U\vec{v}\,' = A(t)U\vec{v} + U\vec{v}\,' = A(t)\vec{y} + U\vec{v}\,'.$$

Hence, $U\vec{v}\,' = \vec{f}(t)$ and

$$\vec{y}(t) = U(t)\int U^{-1}(t)\vec{f}(t)\,dt.$$

Liouville's equation. We prove a theorem of Liouville, which generalizes Abel's identity for the Wronskian.

Theorem 27.4 (Liouville's Theorem). *If $Y'(t) = A(t)Y(t)$ on an interval $t \in I$, then*

$$|Y(t)|' = \mathrm{tr}A(t)|Y(t)|. \tag{27.2}$$

Proof. First, if $|Y(t_0)| = 0$ at a point $t_0 \in I$, then $|Y(t)| = 0$ for all $t \in I$, and we are done. We therefore assume that $|Y(t)| \neq 0$ for all $t \in I$.

Let $Y(t_0) = I$ at a point t_0. That is,

$$Y(t_0) = (y_1(t_0)\ \cdots\ y_n(t_0)) = (E_1\ E_2\ \cdots\ E_n).$$

Here, E_j are the unit coordinate vectors in $\mathbb{R}^n$, that is, the n-vector E_j has 1 in the j-th position and zero otherwise.

We use the derivative formula for the determinant

$$\begin{aligned}|Y(t)|' =& \frac{d}{dt}\det(y_1(t)\dots y_n(t))\\ =& \det(y_1'(t)\ y_2(t)\ \cdots\ y_n(t)) + \det(y_1(t)\ y_2'(t)\ \cdots\ y_n(t)) + \cdots + \det(y_1(t)\ \cdots\ y_n'(t)).\end{aligned}$$

This formula is based on the Laplace expansion formula for determinant, and we do not prove it here. Since

$$y_j'(t_0) = A(t_0)y_j(t_0) = A(t_0)E_j = A_j(t_0),$$

where $A_j(t)$ is the jth column of $A(t)$, evaluating the above determinant formula at $t = t_0$ we obtain

$$\begin{aligned}|Y(t_0)|' &= \det(A_1(t_0)\ E_2\ \cdots\ E_n) + \det(E_1\ A_2(t_0)\ \cdots\ E_n) + \cdots + \det(E_1\ E_2\ \cdots\ A_n(t_0))\\ &= a_{11}(t_0) + a_{22}(t_0) + \cdots + a_{nn}(t_0) = \mathrm{tr}A(t_0).\end{aligned}$$

Thus, (27.2) holds at t_0.

In general, let $C = Y(t_0)^{-1}$. Then $U(t) = Y(t)C$ satisfies

$$U' = A(t)U, \qquad U(t_0) = I.$$

Therefore, by the argument above $|U(t_0)|' = \mathrm{tr}A(t_0)|U(t_0)| = \mathrm{tr}A(t_0)$. Since

$$\frac{d}{dt}(|Y(t)C|) = \frac{d}{dt}(|Y(t)||C|) = |Y(t)|'|C|, \qquad \text{at } t = t_0,$$

it follows that $\mathrm{tr}A(t_0) = |Y(t_0)|'|Y(t_0)|^{-1}$. Since t_0 is arbitrary, the proof is complete. □