18.034 Honors Differential Equations
Spring 2009

LECTURE 23. THE DIRAC DISTRIBUTION

Impulse signals: Dirac's idea. Let $a > 0$ be small. Let $f_a(t)$ be identically zero except for the interval $t \in [0, a]$ and $\displaystyle\int_{-\infty}^{\infty} f_a(t)dt = \int_0^a f_a(t)dt \neq 0$. If the integral is not very small then $f_a(t)$ must be quite large in the interval $t \in [0, a]$, and the function describes the "impulsive" behavior. An "impulsive function" means a signal which acts for a very short time but produces a large effect. The physical situation is exemplified by a lightning stroke on a transmission line or a hammer blow on a mechanical system.

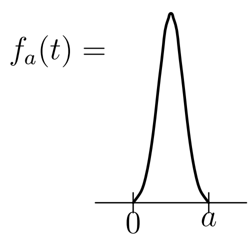

Figure 23.1. Graph of a typical impulsive function

In the early 1930's the Nobel Prize winning physicist P. A. M. Dirac developed a controversial method for dealing with impulsive functions. Let $a \to 0+$. The function $f_a(t)/(\int f_a(t)\, dt)$ approaches to, say, $\delta(t)$ which takes zero for $t \neq 0$ and the integral of $\delta(t)$ over any interval containing 0 is the unity. The function $\delta(t)$ is called "the Dirac delta function".

To formulate the Dirac delta function, let $a > 0$ and let

$$f_a(t) = \begin{cases} 1/a, & t \in [0, a) \\ 0, & \text{elsewhere,} \end{cases}$$

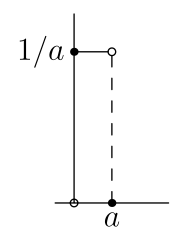

Figure 23. 2. Graph of $f_a(t)$

so that $\displaystyle\int_{-\infty}^{\infty} f_a(t)\, dt = \int_0^a f_a(t)\, dt = 1$ for all $a > 0$.

If $f_a(t) \to \delta(t)$ as $a \to 0+$, then

(23.1) $$\delta(t) = 0 \quad \text{for} \quad t \neq 0, \qquad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

(This is often used as the definition of $\delta(t)$ in elementary differential equations textbooks.)

It is easy to show that

$$\mathcal{L}[f_a(t)] = \int_0^a e^{-st}\frac{1}{a}\, dt = \frac{1 - e^{-sa}}{sa} \to 1 \qquad \text{as} \quad a \to 0.$$

In this sense, $\delta(t)$ describes the effect of

(23.2) $$\mathcal{L}[\delta(t)] = 1.$$

Let us say that no ordinary function with the property (23.1) exists, and whatever else $\delta(t)$ may be, it is not a function of t! However, says Dirac, one formally treats $\delta(t)$ as if it were a function and gets the right answer.

In the late 1940's, the French mathematician Laurent Schwartz* succeeded in placing the delta function on a firm mathematical foundation. He accomplished this by enlarging the class of all functions so as to include the delta function, called it the class of *distributions*.

Here, we first explore the usefulness of $\delta(t)$ in (23.1), and then we indicate its mathematical meaning.

Examples. Let us consider the initial value problem

$$y'' + y = \delta(t), \qquad y(0) = y'(0) = 0.$$

Taking the transform,

$$\mathcal{L}y = \frac{1}{s^2+1}, \qquad y(t) = h(t)\sin t.$$

Here, we use (23.2).

The solution y is continuous for all $t \in (-\infty, \infty)$ and it satisfies the differential equation everywhere except $t = 0$. At $t = 0$, however, it satisfies neither the differential equation nor the initial condition. It is not even differentiable at $t = 0$. Indeed, $y'(0+) = 1$ and $y'(0-) = 0$. The unit impulse signal $\delta(t)$ produces a jump of magnitude 1 in $y'(t)$ at $t = 0$.

Let us now consider

$$y'' + y = f_a(t) = \begin{cases} 1/a, & t \in [0, a) \\ 0, & \text{elsewhere,} \end{cases} \qquad y(0) = y'(0) = 0.$$

We write $f_a(t) = \frac{1}{a}(h(t) - h(t-a))$. Taking the transform then gives

$$\mathcal{L}y = \frac{1}{s^2+1}\frac{1}{a}\frac{1-e^{-sa}}{s}$$

and thus,

$$\begin{aligned} y_a(t) &= \frac{1}{a}h(t)(1-\cos t) - \frac{1}{a}h(t-a)(1-\cos(t-a)) \\ &= \begin{cases} 0, & (-\infty, 0] \\ \dfrac{1-\cos t}{a}, & t \in (0, a) \\ \dfrac{\cos(t-a)-\cos t}{a}, & [a, \infty). \end{cases} \end{aligned}$$

As $a \to 0+$, the second interval $t \in (0, a)$ vanishes, and the third interval $t \in [a, \infty)$ tends to $[0, \infty)$ and the functional expression on this interval tends to $\sin t$. In other words, as $a \to 0$ the solution $y_a(t)$ gives the same solution as was obtained with $\delta(t)$. We note that an uncritical use of $\delta(t)$ as in (23.1) and (23.2) gives the correct answer as obtained by a conventional passage to the limit. Moreover, the conventional method is much more difficult. It is where the usefulness of $\delta(t)$ lies.

Next, let us consider

$$y'' + 2y' + 2y = \delta(t), \qquad y(0) = y'(0) = 0.$$

Taking the transform,

$$\mathcal{L}y = \frac{1}{s^2+2s+2} = \frac{1}{(s-1)^2+1}, \qquad y(t) = h(t)e^{-t}\sin t.$$

This example illustrates another feature of impulsive signals that it induces a lasting effect.

*Laurent Schwartz wass awarded a Fields medal (a mathematical equivalent of the Nobel prize) in 1950 for his creation of the theory of distributions.

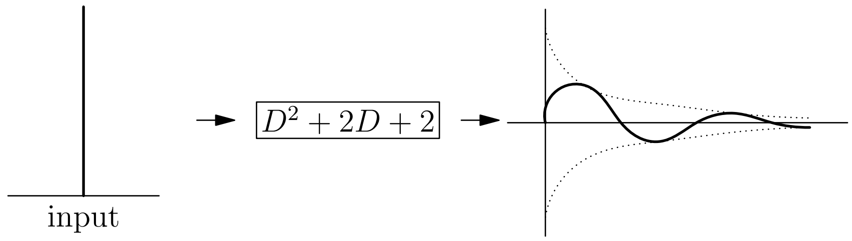

Figure 23.3. The effect of impulsive signals.

Theory of distributions. We conclude the lecture with a very brief description of the germ of Laurent Schwartz's brilliant idea.

A function is characterized by giving its value at each t. A *distribution* $\delta(t)$ is characterized, not by its value at t, but by giving its value $\delta\{\phi\}$ on a suitable class of functions ϕ, called *test functions*. Test functions are assumed to have derivatives of all orders and to vanish outside of a finite interval.

Let us define δ as

$$\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt = \phi(0) \tag{23.3}$$

for any test function ϕ. We note here that we cannot speak of the value of $\delta(t)$ at t. The only meaningful quantity is $\displaystyle\int_{-\infty}^{\infty} \delta(t)\phi(t)\,dt$. The distribution $\delta(t)$ is never used alone, but only in combination with functions.

If $\delta(t)$ were a function and if the integral in (23.3) were an ordinary integral, then by a change of variables,

$$\int_{-\infty}^{\infty} \delta(t-c)\phi(t)\,dt = \int_{-\infty}^{\infty} \delta(t)\phi(t+c)\,dt.$$

By (23.3) the right side gives the value of $\phi(t+c)$ at $t=0$, that is, $\phi(c)$. Now we take it as the definition of the left side. Thus, $\delta(t-c)$ is defined as

$$\int_{-\infty}^{\infty} \delta(t-c)\phi(t)\,dt = \phi(c)$$

for any test function ϕ.

Similarly, if $\delta'(t)$ were a continuously differentiable function, then an integration by parts would yield

$$\int_{-\infty}^{\infty} \delta'(t)\phi(t)\,dt = -\int_{-\infty}^{\infty} \delta(t)\phi'(t)\,dt.$$

By (23.3) the right side is $-\phi'(0)$, and again we take it to define $\delta'(t)$ as

$$\int_{-\infty}^{\infty} \delta'(t)\phi(t)\,dt = -\phi'(0)$$

for any test function ϕ. With the same line of thought, $\delta^{(n)}(t-c)$ is defined as

$$\int_{-\infty}^{\infty} \delta^{(n)}(t-c)\phi(t)\,dt = (-1)^n\phi^{(n)}(c)$$

for any test function ϕ.

For $a<b$, let us define

$$\int_a^b \delta(t-c)\phi(t)\,dt = \int_{-\infty}^{\infty} (h(t-a)-h(t-b))\delta(t-c)\phi(t)\,dt = \begin{cases} \phi(c), & \text{if } a \leqslant c \leqslant b \\ 0, & \text{otherwise.} \end{cases}$$

With the choice $\phi(t) = e^{-st}$, where s is a constant, we obtain

$$\mathcal{L}[\delta(t-c)] = \int_0^\infty e^{-st}\delta(t-c)\,dt = \begin{cases} e^{-sc}, & c \geqslant 0 \\ 0, & c < 0 \end{cases}$$

when $c \geqslant 0$. When $c < 0$ the result is zero. If $c = 0$ then the above gives the formula $\mathcal{L}\delta(t) = 1$, which agrees with (23.2).

These are definitions, but the discussion shows that the definitions are consistent with the ordinary rules of calculus. This is the reason why $\delta(t)$ can be treated as a function of the real variable t even though it is not a function.

Finally, assume that $y(t) = 0$ for $t < 0$ and $y'(t) = \delta(t)$ for $t > 0$. The Laplace transform suggests that $s\mathcal{L}y = 1$, and in turn, y agrees with the Heaviside function $h(t)$ except perhaps at $t = 0$. (At $t = 0$, neither the physics nor the mathematics of the problem is clear.) In this sense,

$$h'(t) = \delta(t).$$