18.034 Honors Differential Equations
Spring 2009

UNIT V: THE LAPLACE TRANSFORM

The method of the Laplace transform enables one to solve initial value problems without going through finding the general solution and then evaluating the arbitrary constants. The method is particularly useful in dealing with discontinuous inputs (closing of a switch) and with impulsive inputs. An integral expression known as the convolution allows an easy passage from the frequency domain (s-domain) to the time domain (t-domain) and leads us to explicit solutions in the time domain. The notions of the transfer function and the pole diagram are of continuing interest in engineering.

LECTURE 19. LAPLACE TRANSFORM

The *Laplace transform* of a function f, defined for $t \in [0, \infty)$, is the function $F(s)$ defined as

$$F(s) = \mathcal{L}[f(t)](s) = \int_0^\infty e^{st} f(t)dt = \lim_{T\to\infty} \int_0^T e^{st} f(t)dt \tag{19.1}$$

provided that the limit exists for all sufficiently large s. That means, there is an s_0, depending on f, such that the limit exists whenever $s > s_0$. The parameter s is in general considered to be complex, but in these notes it is taken to be real.

It is named in honor of Pierre-Simon Laplace, who used the transform in his work on probability theory.

Example 19.1. We compute

$$\lim_{T\to\infty} \int_0^T e^{-st} dt = \lim_{T\to\infty} \frac{1 - e^{-sT}}{s} = \begin{cases} \dfrac{1}{s} & s > 0 \\ \infty & s \leqslant 0 \end{cases}.$$

When $s = 0$, the value of integral is T, which tends to infinity as $T \to \infty$. This shows that $\mathcal{L}[f(t)](s) = 1/s$ for $s > 0$.

Example 19.2. For a a real constant,

$$\lim_{T\to\infty} \int_0^T e^{-st} e^{at} dt = \lim_{T\to\infty} \frac{e^{(a-s)T} - 1}{a - s} = \begin{cases} \dfrac{1}{s-a} & s > a \\ \infty & s \leqslant a \end{cases}.$$

This shows that $\mathcal{L}[e^{at}](s) = 1/(s - a)$ for $s > a$.

If a is a complex constant, the same calculation shows that $\mathcal{L}[e^{at}](s) = 1/(s - a)$ for $s > \text{Re}\, a$.

Exercise. Using (19.1), compute $\mathcal{L}[\cos bt]$ and $\mathcal{L}[\sin bt]$.

ANSWERS. $\dfrac{s}{s^2 + b^2}$ and $\dfrac{b}{s^2 + b^2}$.

Two notations for the Laplace transform will be used here. In the first notation, the connection between a function f and its transform F is indicated by lower case and upper case. In the second notation, the Laplace transform of f is denoted by $\mathcal{L}f$, where $\mathcal{L}$ is the Laplace transform operator.

As a transform, $\mathcal{L}$ is linear. That is,

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2]$$

for all functions f_1 and f_2 whose Laplace transforms exist and for all constants c_1 and c_2.

Functions of exponential type. It is noted that whereas $f(t)$ is defined for $t \in [0, \infty)$, its Laplace transform is usually defined in a different interval. For example, the Laplace trnasform of e^{2t} is only defined for $s \in (2, \infty)$. This is because the integral (19.1) will only exist, in general, for s sufficiently large.

One very serious difficulty with the definition (19.1) is that this integral may fail to exist for every value of s. For example, $f(t) = e^{t^2}$ does not possess a Laplace transform. To guarantee that the Laplace transform of $f(t)$ exists at least in some interval $s \in (s_0, \infty)$, we impose some conditions on $f(t)$.

Definition. A real- or complex-valued function $f(t)$ is said to be of *exponential type*, denoted by $f \in E$, if

(i) on any interval $[0, T]$, the function is defined and piecewise continuous, that is, f is continuous except at finitely many points, and

(ii) $|f(t)| \leqslant Ae^{Bt}$ at all points $t \in [0, \infty)$ where it is defined for some constants A and B depending on f.

Proposition 19.3. *If $f \in E$ then its Laplace transform exists for all s sufficiently large.*

Proof. Since $f(t)$ is piecewise continuous, the integral $\int_0^T e^{-st} f(t) dt$ exists for all T. To prove that this integral has a limit as $T \to \infty$ for all s sufficiently large, we observe that

$$\int_0^T e^{-st} |f(t)| dt \leqslant \int_0^T e^{-st} Ae^{Bt} dt \leqslant \frac{A}{s - B}$$

for $s > B$. Consequently*, the Laplace transform of $f(t)$ exists for $s > B$. □

Solving differential equations. The real usefulness of the Laplace transform in solving differential equations lies in the fact that the Laplace transform of $f'(t)$ is very closely related to the Laplace transform of $f(t)$. This is the content of the following important lemma.

Lemma 19.4. *If f is continuous on $[0, \infty)$ and $f' \in E$ then $f \in E$ and*

$$\mathcal{L}[f'(t)](s) = s\mathcal{L}[f(t)](s) - f(0). \tag{19.2}$$

Proof. Integration by parts gives

$$\int_0^T e^{-st} f'(t) dt = [e^{-st} f(t)]_0^T + s \int_0^T e^{-st} f(t) dt. \tag{19.3}$$

The proofs of $f \in E$ and that integration by parts is permissible under the hypothesis are left as exercises. If $|f(t)| \leqslant Ae^{Bt}$ and $s \geqslant B + 1$ then the first term on the right side of (19.3) at the upper limit T satisfies

$$|e^{-sT} f(T)| \leqslant e^{(B+1)T} Ae^{BT} = Ae^{-T}$$

which tends to zero as $T \to \infty$. Hence letting $T \to \infty$ in (19.3) proves the assertion. □

Exercise. Show that

$$\mathcal{L}[f^{(n)}(t)](s) = s^n \mathcal{L}[f(t)](s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0). \tag{19.4}$$

*Here, we use the calculus fact that if $f(t)$ is piecewise continuous then $\int_0^\infty f(t)dt$ exists if and only if $\int_0^\infty |f(t)|dt$ exists.

The differential equations considered in the notes have solutions y that are sums of functions of the form $ct^k e^{\lambda t}$ with $k = 1, 2, \ldots$. Therefore the solutions as well as their derivatives satisfy the hypothesis of Lemma 19.4. Therefore,

$$\mathcal{L}y' = s\mathcal{L}y - y(0).$$

This relation enables us to transform an initial value problem of a differential equation for y into an algebraic equation for $\mathcal{L}y$, which is much easier to solve. The theory takes its form from a symbolic method developed by the English engineer Oliver Heaviside.

As an example, let us consider the initial value problem

$$y' - y = e^t, \qquad y(0) = 1.$$

Taking the Laplace transform gives

$$\mathcal{L}y' - \mathcal{L}y = \mathcal{L}e^t \quad \text{or} \quad s\mathcal{L}y - 1 - \mathcal{L}y = \frac{1}{s-1}.$$

Hence, $\mathcal{L}y = \dfrac{1}{(s-1)^2} + \dfrac{1}{s-1}$. This tells us the Laplace transform of the solution $y(t)$. To find $y(t)$ we must consult the *inverse Laplace transform*, formally denoted by $\mathcal{L}^{-1}$. Just as $\mathcal{L}y$ is expressed explicitly in terms of $y(t)$, via (19.1), we can write down an explicit formula† for $y(t)$ in terms of $\mathcal{L}y$. However, this formula involves an integration with respect to a complex variable and it is beyond the scope of this course. Therefore, instead, we will derive several properties of the Laplace transform operator in the next lecture which will enable us to invert many Laplace transforms by inspection, that is, by recognizing "which functions they are the Laplace transform of".

The procedure necessitates the following justifications.

Theorem 19.5 (Uniqueness Theorem)**.** *If f and g are functions of class E and their Laplace transforms agree for all large s, then $f(t) = g(t)$ at all points $t \geqslant 0$ where both functions are continuous.*

The proof depends on another theorem which is of independent interest.

Lemma 19.6. *If q is continuous for $0 \leqslant x \leqslant 1$, then*

$$\int_0^1 x^n q(x)dx = 0, \quad \textit{for } n = 0, 1, 2, \ldots \qquad \textit{implies} \qquad q(x) = 0 \quad \textit{for } 0 \leqslant x \leqslant 1.$$

Proof of Theorem 19.5. Given f and g as in the statement of Theorem 1, let u, v, w with their Laplace transforms U, V, W, respectively, be defined by

$$u(t) = f(t) - g(t), \qquad v(t) = \int_0^t u(\tau)d\tau, \qquad w(t) = e^{-ct}v(t).$$

By assumption, then, $U(s) = 0$ for all large s, and our goal is to prove that $u(t) = 0$ at all points of continuity. Note that $V(s) = U(s)/s = 0$ for large s. Accordingly, if $v(t) = 0$ then by the fundamental theorem of calculus follows that $u(t) = 0$ at all points of continuity. Hence, we will work with the continuous function v rather than u.

For the constant c large enough, by the shift theorem of the Laplace transform, $W(s) = V(s + c) = 0$ for all $s > 0$ (not only for large s). Since $u \in E$ follows $v \in E$, and for large c, it yields

(19.5) $$\lim_{t\to\infty} w(t) = 0.$$

†It is given by the so-called *Fourier-Mellin integral*

$$f(t) = \mathcal{L}^{-1}[F(s)](t) = \frac{1}{2\pi i}\int_{\gamma - i\cdot\infty}^{\gamma + i\cdot\infty} e^{st}F(s)ds,$$

where γ is a real number with $\gamma > \mathrm{Re}(s_F)$ and s_F is a singularity of $F(s)$.

By the change of vaiable $x = e^{-t}$ we obtain that

$$W(s) = \int_0^\infty e^{-st} w(t)dt = \int_0^1 x^{s-1} w(-\ln x)dx.$$

Let

$$q(x) = \begin{cases} w(-\ln x) & \text{for} \quad 0 < x \leqslant 1 \\ 0 & \text{for} \quad x = 0 \end{cases}.$$

Equation (19.5) guarantees that q is continuous for $0 \leqslant x \leqslant 1$. Since $W(s) = 0$ for all $s \geqslant 0$, we certainly have $W(s) = 0$ for $s = 1, 2, 3, \ldots$. Then the assertion follows from Lemma 19.6. $\square$

Proof of Lemma 19.6. Suppose $q(x_0) \neq 0$ at some point in $(0, 1)$. We may assume that $q(x_0) > 0$. Continuity then gives positive constants ϵ, δ such that

$$|x - x_0| \leqslant 2\delta \quad \text{implies} \quad q(x) \geqslant \epsilon.$$

Let m be an arbitrary positive integer and define

$$p(x) = 1 + 4\delta^2 - (x - x_0)^2, \qquad I_m = \int_0^1 p^m(x)q(x)dx.$$

By the binomial theorem p^m is a polynomial in x, and hence $I_m = 0$ under the hypothesis.

On the other hand $I_m = J_1 + J_2 + J_3$, where J_1, J_2, J_3 are integrals over the part of $[0, 1]$ in which

$$|x - x_0| < \delta, \qquad \delta \leqslant |x - x_0| \leqslant 2\delta, \qquad |x - x_0| > 2\delta,$$

respectively. On these three intervals we have

$$p(x) \geqslant 1 + 3\delta^2, \qquad p(x) \geqslant 1, \qquad |p(x)| \leqslant 1$$

and also, if M is a sufficiently large constant,

$$q(x) \geqslant \epsilon, \qquad q(x) \geqslant \epsilon, \qquad |q(x)| \leqslant M.$$

It is easy to check that

$$J_1 \geqslant (1 + 3\delta^2)^m \delta\epsilon, \qquad J_2 \geqslant 0, \qquad J_3 \geqslant -M.$$

The first expression tends to infinity as $m \to \infty$, and hence $I_M \to \infty$. This contradicts that $I_m = 0$. $\square$

Example 19.7. Solve the initial value problem

$$y'' + 4y = 0, \qquad y(0) = 5, \quad y'(0) = 6.$$

SOLUTION. Taking the transform gives

$$s^2 \mathcal{L}y - 5s - 6 + 4\mathcal{L}y = 0.$$

Hence,

$$\mathcal{L}y = \frac{5s + 6}{s^2 + 4} = 5\frac{s}{s^2 + 4} + 3\frac{2}{s^2 + 4},$$

and $y(t) = 5 \cos 2t + 3 \sin 2t$.