18.034 Honors Differential Equations
Spring 2009

LECTURE 14. STABILITY

The notion of stability. Roughly speaking, a system is called *stable* if its long-term behavior does not depend on significantly the initial conditions.

An important result of mechanics is that a system of masses attached in (damped or undamped) springs is stable. A similar result is in network theory. In these notes, we study the differential equation of the form

$$y'' + py' + qy = f(t), \tag{14.1}$$

where p, q are constatnts and $f(t)$ represents the external forces.

We learned that the general solution of (14.1) has the form

$$y = c_1y_1 + c_2y_2 + y_p, \tag{14.2}$$

where c_1, c_2 are arbitrary constants and y_p is a particular solution of (14.1); $c_1y_1 + c_2y_2$ is the complementary solution, that is, the general solutions of the homogeneous equation (14.1) with $f(t) = 0$.

The initial conditions determine the values of c_1 and c_2. Thus, we say the system (14.1) is stable if $c_1y_1 + c_2y_2 \to 0$ as $t \to \infty$ for any coice of c_1 and c_2.

If (14.1) is stable then y_p is called the *steady-state solution* and $c_1y_1 + c_2y_2$ is called *transient*. Physically, in a stable system, the output is the sum of a transient term, which depends on the initial conditions, but whose effects die out over time, and a steady-state, which represents the response of the system to the input $f(t)$ after a long time.

Stability conditions. We study under what circumstances the differential equation $Ly = f$, where

$$L = D^n + p_1D^{n-1} + ... + p_{n-1}D + p_n, \tag{14.3}$$

where p_j are constants, is stable.

Definition 14.1. The differential equation $Ly = f$, where L is given in (14.3) is called:

(i) *asymptotically stable* if every solution of $Ly = 0$ tends to zero as $t \to \infty$;
(ii) *stable* if every solution of $Ly = 0$ remains bounded a $t \to \infty$;
(iii) *unstable* if it is not stable.

We note that stability concerns only the behavior of the solutions of the corresponding homogeneous equation $Ly = 0$.

When $f(t) = 0$, then a steady-state solution is $y \equiv 0$. In this case, the system is stable if small initial departures from the steady-state remain small with the lapse of time.

By definition, $Ly = f$ is asymptotically stable if every basis solution of $Ly = 0$ tends to zero as $t \to \infty$ and it is stable if the basis solutions remain bounded. In view of the characteristic polynomial of L and the fundamental theorem of algebra, we write

$$L = (D - \lambda_1)^{k_1}(D - \lambda_2)^{k_2} \cdots (D - \lambda_m)^{k_m},$$

where $\lambda_j \in \mathbb{C}$ are all distinct and $k_1 + k_2 + ... + k_m = n$.

Exercise. The general solution of the homogeneous equation $Ly = 0$ is given by

$$y(t) = c_1(t)e^{\lambda_1 t} + c_2(t)e^{\lambda_2 t} ...... + c_m(t)e^{\lambda_m t},$$

where $c_j(t)$ is an arbitrary polynomial of degree $k_j - 1$.

Exercise. If r is a nonnegative integer and $\lambda \in \mathbb{C}$, show that

$$\lim_{t\to\infty} |t^r e^{\lambda t}| = 0 \qquad \text{if} \quad \text{Re}\lambda < 0.$$

Therefore, $Ly = f$ is asymptotically stable if $\text{Re}\lambda_j < 0$ for all j, and it is stable if $\text{Re}\lambda_j < 0$ or $\text{Re}\lambda_j = 0$ and $k_j = 1$.

We summarize the result.

Theorem 14.2. *The differential equation $Ly = f$ is asymptotically stable if every root of the characteristic polynomial of L has a negative real part, and it is stable if every multiple root has a negative real part and no simple root has a positive real part.*

Example 14.3. We consider the second-order differential equation

$$y'' + py' + qy = 0, \qquad p, q \text{ are constants.} \tag{14.4}$$

We recall that the discriminant $\Delta = p^2 - 4q$ tells us about the nature of the solutions, and hence about the stability of (14.4)

If $q < 0$ then $\Delta > 0$ and the characteristic polynomial $\lambda^2 + p\lambda + q$ has two real roots with opposite signs. Therefore, (14.4) is unstable.

If $p < 0$ then at least one root of the characteristic polynomial must have a positive real part. Hence, (14.4) is unstable.

If $p = 0$ and $q > 0$, then (14.4) reduces to $y'' + qy = 0$ with $q > 0$. Hence, it is stable but asymptotically stable.

Finally, let $p > 0$ and $q > 0$. If $\Delta \leqslant 0$ then the roots of the characteristic polynomial have negative real parts, and (14.4) is asymptotically stable. If $\Delta > 0$ then $\Delta = p^2 - 4q < p^2$ and thus $\sqrt{\Delta} < p$. Therefore, (14.4) is asymptotically stable.

In summary, (14.4) is asymptotically stable if and only if $p > 0$ and $q > 0$, and stable if and only if $p \geqslant 0$ and $q > 0$.

Stability of higher-order differential equations. The above example phrases the stability criterion for (14.4) in terms of the coefficients of the equation. This is convenient since it does not require one to calculate the roots of the characteristic polynomial.

For higher-order equations,

$$y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y = 0, \qquad p_j \text{ are constants,} \tag{14.5}$$

it is not too hard to show that if (14.5) is asymptotically stable then $p_j > 0$ for all j (Exercise). But, the converse is not true (Exercise). For the implication of a criterion for coefficients of (14.5) for stability, the coefficients must satisfy a more complicated set of inequalities, which we state without proof in the following.

Routh-Hurwitz Criterion for Stability. The differential equation (14.5) is asymptotically stable if and only if in the determinant

$$\begin{vmatrix} p_1 & 1 & 0 & 0 & \dots & 0 \\ p_3 & p_2 & p_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{2n-1} & p_{2n-2} & \cdots & \cdots & \cdots & p_n \end{vmatrix},$$

where $p_k = 0$ if $k > n$, all of its n principal minors, that is, the subdeterminants in the upper left corner having sizes respectively $1, 2, \dots, n$,

$$p_1, \qquad \begin{vmatrix} p_1 & 1 \\ p_3 & p_2 \end{vmatrix}, \qquad \begin{vmatrix} p_1 & 1 & 0 \\ p_3 & p_2 & p_1 \\ p_5 & p_4 & p_3 \end{vmatrix}, \qquad \dots$$

are positive.

Exercise. We consider

$$(D^4 + 2D^3 + 6D^2 + 5D + 2)y = 260 \sin 2t.$$

(a) Find a particular solution. (Answer. $11 \cos 2t - 3 \sin 2t$.)

(b) Show that the corresponding characteristic polynomial is factorized as

$$p(\lambda) = (\lambda^2 + 3\lambda + 2)(\lambda^2 + \lambda + 1),$$

and hence the zeros have negative real parts.

(c) Show that the determinant

$$\begin{vmatrix} 4 & 1 & 0 & 0 \\ 5 & 6 & 4 & 1 \\ 0 & 0 & 0 & 2 \end{vmatrix}$$

satisfies the Routh-Hurwitz criterion.