18.034 Honors Differential Equations
Spring 2009

LECTURE 13. INHOMOGENEOUS EQUATIONS

We discuss various techniques for solving inhomogeneous linear differential equations.

Variation of parameters: the Lagrange procedure. Let us consider the linear second-order differential operator

$$Ly = y'' + p(t)y' + q(t)y \tag{13.1}$$

with variable coefficients.

If a nonvanishing solution of a homogeneous equation $Ly = 0$ is known, then the corresponding inhomogeneous equation $Ly = f$ can be solved, in general, by two integrations. It was discovered by Lagrange that if two linearly independent solutions of $Ly = 0$ are known, then the inhomogeneous equation $Ly = f$ can be solved by a *single* integration.

Let u and v be a pair of linearly independent solutions of $Ly = 0$, and form the expression

$$y = au + bv. \tag{13.2}$$

If a and b are constant, this represents the general solution of $Ly = 0$. We will the inhomogeneous equation $Ly = f$ by choosing a trial solution of this form, but with a and b functions of t, rather than constants. The method is called the method of *variation of parameters.*

Let a and b are differentiable functions of t. By differentiation,

$$y' = (au' + bv') + (a'u + b'v).$$

We require

$$a'u + b'v = 0 \tag{13.3}$$

so that $y' = au' + bv'$. This simplifies the calculation of the second derivative, and

$$y'' = (au'' + bv'') + (a'u' + b'v').$$

Therefore,

$$Ly = y'' + py' + qy = aLu + bLv + a'u' + b'v' = a'u' + b'v'.$$

The second equality uses that $Lu = Lv = 0$.

Solving $Ly = f$ in the form in (13.2) then reduces to the linear system

$$a'u + b'v = 0,$$
$$a''u' + b'v' = f,$$

in the unknown a' and b'. In the matrix form,

$$\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}.$$

By Cramers rule, we solve the system, and

$$a' = \frac{\begin{vmatrix} 0 & v \\ f & v' \end{vmatrix}}{\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}}, \qquad b' = \frac{\begin{vmatrix} u & 0 \\ u' & f \end{vmatrix}}{\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}}.$$

Here, the notation $|\cdot|$ stands for the determinant of the matrix. The denominator is the Wromskian $W(u,v)$, so that we may write them as

$$a' = \frac{-fv}{W(u,v)}, \qquad b' = \frac{fu}{W(u,v)}.$$

Finally, by integration, we obtain the *Lagange formula*

$$y(t) = u(t)\int \frac{-fv}{W(u,v)}dt + v(t)\int \frac{fu}{W(u,v)}dt. \tag{13.4}$$

Lagrange's procedure extends to equations of order n and it represents an important advance in the theory of differential equations.

A similar idea already appeared. For example, when studying the linear first-order differential equations, we replaced the homogeneous solution ce^P by ve^P, where v is a function.

Example 13.1. Consider the Euler equation

$$x^2y'' - 2xy' + 2y = x^2 f(x), \qquad x > 0, \tag{13.5}$$

where the prime denotes the differentiation in the x-variable.

By the technique discussed in the previous lecture, we compute

$$u = x, \qquad v = x^2, \qquad W(u,v) = x^2.$$

For $x > 0$, we write (13.5) as

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = f(x).$$

Then Lagrange's formula (13.4) gives

$$y(x) = -x\int f(x)dx + x^2\int \frac{f(x)}{x^2}.$$

Exercise. If $f(x) = x^m$, where m is a constant, in the above example, show that a particular solution of (13.5) is

$$y_p(x) = \begin{cases} -x\log x & \text{if } m = -1, \\ x^2\log x & \text{if } m = 0, \\ \dfrac{x^{m+2}}{m(m+1)} & \text{otherwise.} \end{cases}$$

The general solution of (13.5) is $y(t) = c_1x + c_2x^2 + y_p(x)$.

The Green's function: initial value problems. As an important application of the formula (13.4) we can find an integral representation of the initial value problem for $Ly = f$, where L is given in (13.1).

Let t_0 be a point on the interval I. Integrating (13.4) from t_0 to t,

$$\begin{aligned} y(t) =& u(t)\int_{t_0}^t \frac{-f(t')v(t')}{W(t')}dt' + v(t)\int_{t_0}^t \frac{f(t')u(t')}{W(t')}dt' \\ =& \int_{t_0}^t \frac{u(t')v(t) - u(t)v(t')}{u(t')v'(t') - u'(t')v(t')}f(t')dt' \end{aligned}$$

This function satisfies the conditions

$$y(t_0) = 0, \qquad y'(t_0) = 0.$$

Indeed, $y(t) = a(t)u(t) + b(t)v(t)$ and $y'(t) = a(t)u'(t) + b(t)v'(t)$ where $a(t) = \dfrac{-f(t')v(t')}{W(t')}dt'$ and $b(t) = \dfrac{f(t')u(t')}{W(t')}dt'$.

In summary, the function defined as

$$y(t) = \int_{t_0}^{t} G(t,t')f(t')dt', \qquad G(t,t') = \frac{u(t')v(t) - u(t)v(t')}{u(t')v'(t') - u'(t')v(t')}$$

solves the initial value problem,

$$Ly = f, \qquad y(t_0) = 0, \quad y'(t_0) = 0.$$

The function $G(t,t')$ is called the *Green's function*.

Example 13.2. We continue studying the Euler equation (13.5) satisfying the initial conditions

$$y(x_0) = 0, y'(x_0) = 0 \qquad \text{for some } x_0 > 0.$$

The solution has an integral representation

$$y(x) = x\int_{x_0}^{x}(x-t)\frac{f(t)}{t}dt.$$

For example, if $f(x) = x\sin x$ then

$$y(x) = x\int_{x_0}^{x}(x-t)\sin t dt = x(x-x_0)\cos x_0 - x(\sin x - \sin x_0).$$

Exercise. (The Green's function: boundary value problem) We consider the boundary value problem

$$y'' + p(t)y' + q(t)y = f(t) \quad \text{on } (t_1,t_2), \qquad y(t_1) = y(t_2) = 0.$$

If u and v are linearly independent solutions of the homogeneous equation $y'' + py' + qy = 0$, then show that the solution of the boundary value problem is given by

$$y(t) = \int_{t_1}^{t_2} G(t',t)f(t')dt',$$

where $G(t',t) = \begin{cases} \dfrac{u(t')v(t)}{W(t')} & \text{if } t_1 \leqslant t' \leqslant t, \\ \dfrac{u(t)v(t')}{W(t')} & \text{if } t \leqslant t' \leqslant t_2. \end{cases}$

The method of annihilators. We introduce another method of finding a particular solution of linear inhomogeneous differential equation with constant coefficients. Let

$$Ly = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1}y' + p_n y,$$

where p_j are real constants. We study the differential equation $Ly = f$, where f is a sum of functions of type

$$t^r e^{\lambda t}, \qquad t^r e^{\mu t}\sin\nu t, \qquad t^r e^{\mu t}\cos\nu t.$$

Note that these functions arise as basis solutions of linear homogeneous differential equations with constant coefficients. We find a differential operator A satisfies $Af = 0$, then we reduce solving $Ly = f$ to solving the homogeneous equation $LAy = 0$. Such an operator A is called an *annihilator* of f.

We illustrate with an example.

Example 13.3. We consider the differential equation

$$y'' - 5y' - 6y = te^t. \tag{13.6}$$

Let $L = D^2 - 5D - 6 = (D-2)(D-3)$. Then (13.6) is written as $Ly = te^t$.

By the exponential shift law for D, we recognize that te^t is a solution of the differential equation $(D-1)^2 y = 0$. In other words, $(D-1)^2$ is an annihilator of te^t. Applying $(D-1)^2$ in (13.6), we obtain the homogeneous differential equation

$$(D-2)(D-3)(D-1)^2 y = 0.$$

It is easy to see that $e^t,\ te^t,\ e^{2t}, e^{3t}$ form a basis of solutions of the above equation. Hence, we set a solution of (13.6) as

$$y(t) = c_1 e^t + c_2 te^t + c_3 e^{2t} + c_4 e^{3t},$$

and determine the constants c_j.

Since $Le^{2t} = 0$ and $Le^{3t} = 0$, moreover, we may set $c_3 = c_4 = 0$. Hence,

$$y(t) = c_1 e^t + c_2 te^t.$$

We compute

$$Ly = (D^2 - 5D - 6)(c_1 e^t + c_2 te^t) = (2c_1 - 3c_2)e^t + 2c_2 te^t = te^t$$

to obtain $c_1 = 3/4$ and $c_2 = 1/2$. Therefore, a particular solution of (13.6) is $y(t) = 3/4e^t + 1/2te^t$.