18.034 Honors Differential Equations
Spring 2009

LECTURE 8. UNIQUENESS AND THE WRONSKIAN.

Differential inequality and uniqueness. We prove the uniqueness theorem for linear second-order differential equations with variable coefficients.

Theorem 8.1 (Uniqueness Theorem)**.** *If $p(t)$ and $q(t)$ are continuous on an open interval I containing t_0, then at most one solution of*

$$y'' + p(t)y' + q(t)y = f(t) \tag{8.1}$$

on I satisfies the initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_1$.

Proof. Let y_1 and y_2 be any two solutions of (8.1) which satisfy the initial conditions. Let $v = y_1 - y_2$. Then,

$$v'' + p(t)v' + q(t)v = 0 \quad \text{on } I \quad \text{and} \quad v(t_0) = v'(t_0) = 0. \tag{8.2}$$

We shall show that $v(t) = 0$ for all $t \in I$.

We consider the function $E(t) = v^2 + (v'(t))^2$. It is readily seen that $E(t) \geqslant 0$ and $E(t_0) = 0$. By differentiating, we obtain

$$\begin{aligned} E'(t) =& 2v(t)v'(t) + 2v'(t)v''(t) = 2v'(t)(v(t) + v''(t)) \\ =& 2v'(t)(v(t) - p(t)v'(t) - q(t)v(t)) \\ =& -2p(t)(v'(t))^2 + 2(1 - q(t))v(t)v'(t). \end{aligned}$$

The second equality uses (8.2). By the Cauchy-Schwartz inequality, then

$$(1 - q(t))v(t)v'(t) \leqslant (1 + |q(t)|)(v^2(t) + (v'(t))^2),$$

whence

$$E'(t) \leqslant (1 + |q(t)|)v^2(t) + (1 + |q(t)| + 2|p(t)|)(v'(t))^2 \leqslant KE(t),$$

where $K \geqslant 1 + \max\limits_{t \in I}(|q(t)| + 2|p(t)|)$ is a constant.

We claim that $E(t) = 0$ for all $t \in I$. Suppose, on the contrary, that $E(t_1) > 0$ at some point t_1. Assume $t_1 > t_0$. The other case can be treated similarly. We compute

$$\frac{d}{dt}(e^{-Kt}E(t)) = e^{-Kt}(E'(t) - KE(t)) \leqslant 0.$$

Hence, $e^{-Kt}E(t)$ is a decreasing function of t. In particular,

$$e^{-Kt_1}E(t_1) \leqslant e^{-Kt_0}E(t_0) = 0.$$

However, $E(t_1) \leqslant 0$, which leads to a contradiction. This completes the proof. □

The above method applies to a broad class of linear and nonlinear differential equations. It applies when y is a complex solution and when $p(t)$ and $q(t)$ are merely bounded.

The Wronskian. The *Wronskian*[*] of two differentiable functions u and v is, by definition,

$$W(u,v;t) = \begin{vmatrix} u(t) & v(t) \\ u'(t) & v'(t) \end{vmatrix} = u(t)v'(t) - u'(t)v(t). \tag{8.3}$$

We write $W(t)$ or $W(u,v)$ to emphasize dependence on t or on the functions.

In the study of a linear differential equation

$$y'' + p(t)y' + q(t)y = 0, \tag{8.4}$$

where p, q are continuous, the Wronskian can be computed easily by the following result.

Theorem 8.2. *(Abel's identity[†]) Let u and v be solutions of* (8.4)*, then the Wronskian $W(u,v;t)$ satisfies the first-order differential equation*

$$W' + p(t)W = 0. \tag{8.5}$$

Consequently,

$$W(u,v;t) = W(u,v,t_0)\exp\left(-\int_{t_0}^{t} p(s)ds\right).$$

Proof. By differentiating $W'(u,v) = uv'' - u''v$. The assertion follows upon substituting u'' and v'' by (8.4) and by cancellation. □

Corollary 8.3. *The Wronskian of two solutions of* (8.4) *is either identically positive, identically negative or identically zero.*

The Wronskian and linear dependence. A collection of functions $u_1, \cdots, u_n$ is called *linearly independent* on the interval I if

$$c_1u_1(t) + \cdots + c_nu_n(t) = 0 \text{ on } t \in I \quad \text{implies} \quad c_1 = c_2 = \cdots = c_n = 0.$$

It is called *linearly dependent* otherwise. If u and v are linearly dependent, then u and v are proportional.

The Wronskian gives a simple criterion for linear dependence.

Lemma 8.4. *Let u and v be differentiable functions on an interval I.*

(i) *If u and v are linearly dependent, then $W(u,v;t) = 0$ for all $t \in I$.*
(ii) *If $W(u,v;t) = 0$ on I and $v \neq 0$, then u and v are linearly dependent.*

The condition $W(u,v) = 0$ on an interval, in general, does not ensure that u and v are linearly dependent. For example, $W(t^3, |t|^3) \equiv 0$ but t^3 and $|t|^3$ are linearly independent on any open interval containing zero.

If u and v are solutions of a linear second-order differential equation, then a stronger result than (ii) in the above lemma holds true.

Theorem 8.5. *Let u and v be solutions of* (8.4)*, where p, q are continuous functions on an interval I.*

If $W(u,v;t_0) = 0$ at some point $t_0 \in I$, then u and v are linearly dependent and hence $W(u,v;t) = 0$ for all $t \in I$. If u and v are linearly independent then $W(u,v;t) = 0$ at no point of I.

Proof. If $W(u,v;t_0) = 0$ then two vectors $(u(t_0), u'(t_0)$ and $(v(t_0), v'(t_0))$ are linearly dependent. Hence, one can choose c_1 and c_2, both cannot be zero, such that

$$c_1u(t_0) + c_2v(t_0) = 0,$$
$$c_1u'(t_0) + c_2v'(t_0) = 0.$$

[*]It is named after the Polish mathematician Józef Hoene-Wroński. He introduced determinants of this form in 1811.
[†]Discovered by the Norvegian mathematician Hentik Abel in 1826

We consider the function $y(t) = c_1u(t) + c_2v(t)$. Since y is a linear combination of u and v, it solves (8.4). Moreover, it satisfies the initial condition $y(t_0) = y'(t_0) = 0$. By the uniqueness theorem, then, $y(t) = 0$ for all $t \in I$. That means, u and v are proportional on I, and it proves the first assertion. The second assertion then is an obvious consequence of the first. □

The fact that (8.4) has no singular points is vital in the above theorem. For example, t^2 and t^3 are linearly independent solutions of the differential equation

$$t^2y'' - 4ty' + 6y = 0.$$

But, $W(t^2, t^3) = t^4$ vanishes at $t = 0$.

The Wronskian has an interesting application of finding a basis of solutions and a particular solution of a linear second-order differential equation.

Theorem 8.6. *Let u be a non-vanishing solution of the differential equation* (8.4).

(i) *The second solution v of* (8.4), *independent of u, is given by*

$$v(t) = cu(t)\int \frac{e^{-P(t)}}{u^2(t)}, \qquad c \neq 0, \tag{8.6}$$

where $P(t) = \int p(t)dt$.

(ii) *a particular solution of the inhomogeneous equation*

$$y'' + p(t)y' + q(t)y = f(t)$$

is given by $w = uz$, where

$$(e^Pu^2z')' = ue^Pf, \qquad P(t) = \int p(t)dt.$$

Proof. (i) We compute

$$\left(\frac{v}{u}\right)' = \frac{uv' - u'v}{u^2} = \frac{W(u,v)}{u^2}.$$

The assertion then follows upon integration and the use of the Abel's identity.

(ii) Substituting $w = uz$ into the equation, we obtain

$$uz'' + (2u' + pu)z' = f.$$

This is a first-order linear differential equation for z'. It is straightforward to compute the integrating factor ue^P. Multiplying the above equation by the factor,

$$z''u^2e^P + e^P(2uu' + pu^2)z' = ue^Pf.$$

This proves the assertion. □

Example 8.7. The trial solution $y = t^m$ shows that the equation

$$t^2y'' - 13ty' + 49y = 0, \qquad t > 0 \tag{8.7}$$

has a solution $u = t^7$. To find a second solution, linearly independent of u, we compute

$$p(t) = \frac{-13}{t}, \qquad P(t) = -13\log t, \qquad e^{-P(t)} = t^{13}.$$

The above theorem then gives

$$v = t^7\int t^{13}t^{-14}dt = t^7\log t.$$

The general solution to (8.7) is therefore

$$t^7(c_1 + c_2\log t),$$

where c_1, c_2 are arbitrary constants.

Next, we consider the inhomogeneous equation

$$t^2 y'' - 13ty' + 49y = t^2 f(t), \qquad t > 0,$$

or

$$y'' - \frac{13}{t}y' + \frac{49}{t^2}y = f(t), \qquad t > 0.$$

Take $u = t^7$, and by the above theorem the particular solution is $w = uz$, where

$$z' = \frac{1}{t}\int \frac{f(t)}{t^6}dt.$$

For example, if $f(t) = t^m$, then

$$w(t) = \begin{cases} \dfrac{t^{m+2}}{(m-5)^2}, & m \neq 5 \\ \frac{1}{2}t^7(\log t)^2, & m = 5. \end{cases}$$